# Trigonometry Teacher's Edition - Teaching Tips

CK-12 Foundation

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# Chapter 1

# Trigonometry TE - Teaching Tips

## 1.1 Trigonometry and Right Angles

This Trigonometry Teaching Tips FlexBook is one of seven Teacher's Edition FlexBooks that accompany the CK-12 Foundation's Trigonometry Student Edition.

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### Basic Functions

#### The Basics of Functions

Some students' eyes may glaze over at the equation used in the "gas mileage" situation, especially if it's been a while since their last algebra class. Watch for students who look confused and walk them through setting up the equation if necessary.

Another way to explain the definition of **domain** is that it is the set of real $x-$values you can plug into the function that will produce a real number as the output. You can show that the domain of $y = 3x$ is the set of all real numbers because any real number $x$ can be multiplied by 3 to get a real number $y$; then you can show that the domain of $y = \sqrt{x}$ is restricted to the nonnegative real numbers by demonstrating that you get a real value for $y$ when you plug in a positive or zero value for $x$, but not when you plug in a negative value.

Similarly, you can explain that the **range** of a function is the set of $y-$values that could be the possible outputs of the function given a real number $x$ as the input.

The "rounding" example provides a good opportunity to remind students that rounding "up," in the case of a negative number, means rounding *toward* zero rather than *away from* zero; since $-5$ is *greater than* $-6$, $-5.5$ should be rounded up to $-5$.

#### Families of Functions

In the "families of functions" table, you may need to clarify what "these functions have a highest exponent of 2" means—i.e., the highest power that appears anywhere in the equation is 2. (We can also say that the degree of the function is 2, or that the function is of the "second degree.") The same holds, of course, for "these functions have a highest exponent of 3," but you also may want to clarify "the ends of the graph have

opposite behavior": it simply means that one end of the graph goes up while the other goes down. You may also want to graph $y = x^3$ to show why it has no local maximum or minimum.

And finally, for students who may have forgotten how asymptotes work, it's worth reminding them that the values of the function approach the asymptote but never actually reach it. Specifically, if a function has a horizontal asymptote, it means that y will get closer and closer to that value as $x$ approaches infinity, but will never quite reach it; if a function has a vertical asymptote, it means that $y$ will get closer and closer to infinity as $x$ gets closer to the given value, but the function is undefined when $x$ is exactly equal to that value.

A useful way to explain direct and inverse variation is that with direct variation, the dependent variable increases when the independent variable increases, while with inverse variation, the dependent variable decreases when the independent variable increases. This makes the contrast between the two types of function clearer.

You can clarify the definition of a periodic function by explaining that all the values of the function repeat themselves every $p$ units. It may be useful to demonstrate with the "weather" example above: $p$ in this case is 12, so you can show that $f(14) = f(2), f(15) = f(3)$, and so on.

### Points to Consider

(You may want to go over these as a group each lesson.)

Using a calculator to graph functions is quicker and more accurate than doing it by hand, but it can be hard to see precisely where the important points on the graph are.

## Angles in Triangles

### Similar Triangles

It may be useful to note that the proportions that show that the ratios of corresponding sides are equal can be derived directly from the proportions that show that the side ratios within the two triangles are equal. (For example, $\frac{AB}{DE} = \frac{BC}{EF}$ can be derived directly from $\frac{AB}{BC} = \frac{DE}{EF}$ .)

If you want to move through example 4 a little more quickly, you can point out that the side lengths in the second triangle are simply half those in the first.

Students may ask whether ASA and SAA are also criteria for determining if two triangles are similar, since criteria like those exist for determining if triangles are congruent. Explain that they are, but for a very simple reason: if two of the angles are congruent, then the third angle must also be congruent, and so the ASA and SAA cases simply reduce to the AAA case.

The HL case, on the other hand, is a special example of SSA: two of the sides are proportional, and an angle that is not between them is congruent. Normally, this would not be enough to determine that two triangles are similar, but what helps us here is that we are dealing with right triangles, which means that we can use two side lengths to determine the third. Once we've found that the other leg is proportional too, then instead of looking at this as a case of SSA, we can see it as a case of SSS: all three pairs of sides are proportional. (Incidentally, we could also see it as a case of SAS: the two legs are proportional, and the angle between them (the right angle) is congruent.)

### Points to Consider

The answer to question #1 can be demonstrated in more than one way. First, you can draw two right angles with the included side between them, and show that the other two sides are now parallel, meaning that they can't ever meet to make a triangle.
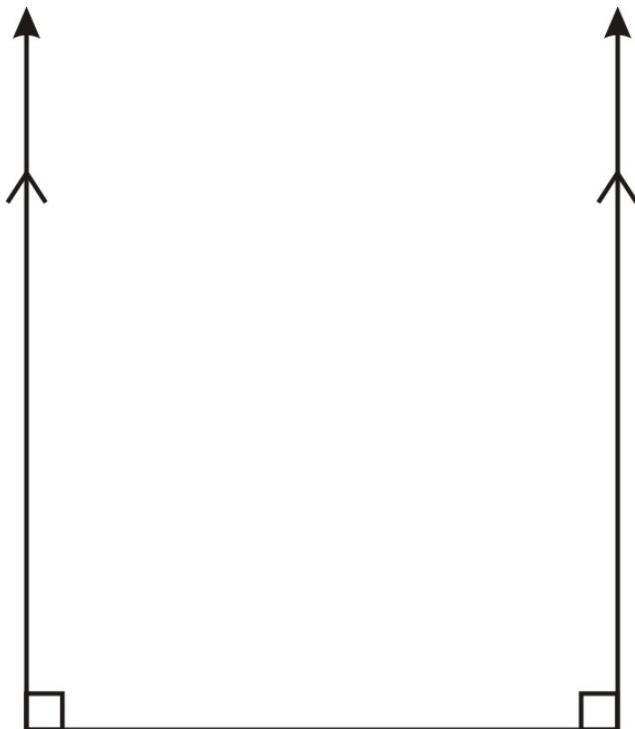
Second, you can point out that two right angles add up to 180°, and since that is the sum of all three angles of a triangle, that would mean the third angle would have to measure 0°, which is not possible.

Similar reasoning holds for question #2. Drawing two obtuse angles with the included side between them demonstrates even more clearly that the other two sides could never meet, and adding together two angle measures greater than 90° each would give you a sum greater than 180°, which is impossible even before you consider the measure of the third angle.

Question #3 has a different answer depending on what situation you are considering. If you just look at an angle in isolation, then in a sense it cannot have a measure greater than 180°, because an angle of, say, 200° could just as easily be described as an angle of 160° viewed from the other side. However, when you are measuring angles of rotation, as the next lesson will cover, an angle can measure more than 180º or even more than 360°. Mentioning this might be a good segue to the next lesson.

## Measuring Rotation

### Measuring Angles

Some students may need to see example 3 worked out in more detail. You may need to spell out that the circumference of the smaller wheel is   meters and that of the larger is $2\pi$ meters; then, rather than simply explaining that the larger wheel rotates once every time the smaller one rotates twice (and therefore rotates twice when the small one rotates four times), you may need to show that the four rotations of the smaller wheel cause its circumference to travel 4 meters along the larger wheel, and that this is equal to $\frac{4\pi}{2\pi}$ or 2 rotations of the larger wheel.

### Angles of Rotation in Standard Position

Students may not have encountered the terms "initial side" and "terminal side" before. Explain, if necessary, that these terms are specific to this particular situation; when we place an angle in standard position, the

initial side is just what we call the side we chose to place along the $x$−axis, and the terminal side is simply the other side.

### Co-terminal Angles

In working through the next example, you may want to take a moment to remind students which quadrant is which (quadrants I through IV proceed counterclockwise starting from the upper right). Knowing the quadrants will be important in upcoming lessons.

Another way to generate the angle $-315°$, of course, is to subtract $315°$ from $360°$ to get $45°$. This method can be faster than rotating clockwise, but students should familiarize themselves with both techniques.

### Points to Consider

Real-life instances of angles of rotation might include a wheel, a swinging door, a doorknob, or a screw.

### Review Questions

You may need to walk students through problem 7. First they must find the total distance the car's inner wheel travels, which is a quarter (90 degrees' worth) of the circumference of a circle with a 100m radius. Then they must find the number of rotations the inner wheel makes in traveling that distance, which takes two steps: first find the circumference of the wheel based on the given diameter of .6m, and then divide the total distance traveled by the circumference of the wheel to find the number of rotations it makes. Next, they must find the distance the outer wheel travels. Since the wheels are 2m apart, the outer wheel follows a curve with radius 2m greater than the curve the inner wheel follows, so the distance it travels is a quarter of the circumference of a circle with a 102m radius. Then, dividing that distance by the circumference of the wheel (already found) gives the number of rotations the outer wheel makes. Finally, they must subtract to find how many more rotations the outer wheel makes than the inner.

## Defining Trigonometric Functions

### The Sine, Cosine, and Tangent Functions

Another way to explain the domain and range of the first three trigonometric functions is as follows: The trigonometric functions take angles as their input, and their output consists of particular ratios of side lengths.

The mnemonic SOH CAH TOA (Sin: Opposite/Hypotenuse; Cosine: Adjacent/Hypotenuse; Tangent: Opposite/Adjacent—pronounced roughly "soak a toe-a") may help students remember the ratios. Another mnemonic is "SCoTt, Oscar Has A Heap Of Apples"—that is, for Sin, Cos, and Tan respectively, the ratios are $\frac{O}{H}, \frac{A}{H}$, and $\frac{O}{A}$.

It may be worth stressing that $\sin(x), \cos(x)$, and $\tan(x)$ are abbreviations for types of functions, and do not indicate that anything is being multiplied.

### Secant, Cosecant, and Cotangent Functions

The fact that the secant, cosecant, and cotangent are reciprocals of the cosine, sine, and tangent functions respectively will be made explicit in a later section of the text, but it may be useful to point it out now, as this may make it easier for students to remember those ratios.

### Trigonometric Functions of Angles in Standard Position

In example 4, you may need to clarify that although the two legs of the triangle in the diagram are labeled 3 and 4, the $x$−coordinate we are working with is actually $-3$, and so when finding the values of the trig functions, we must plug in $-3$ as the length of that leg. (This is where trig functions of angles of rotation start to differ from trig functions of angles in right triangles.)

**Points to Consider**

The Pythagorean Theorem is useful in trigonometry in at least two ways: it helps us find the third side of a right triangle when we need to, and it helps establish some important trigonometric identities. However, the latter won't be covered for a couple more lessons, so you may or may not want to even mention it at this point.

Values of trig functions can be negative when we are dealing with angles of rotation instead of angles in right triangles, because we define the functions in a slightly different way to allow us to describe many more cases. Angles in right triangles must be less than $90°$, and when we work out the trig functions for those angles, we always get positive numbers because the triangles' side lengths are always positive. But when we define the trig functions by reference to $x-$ and $y-$coordinates and the unit circle, we now have a way of finding their values for angles greater than $90°-$and it turns out that some of those values are negative, because the $x-$ and $y-$coordinates we use to find them are sometimes negative. Similarly, trig values can be undefined when we try to find them for quadrantal angles, because some of the coordinates of those angles equal zero.

(All of this will be covered in more detail in the next lesson.)

The unit circle is useful because it gives us an easy way to calculate the trig functions for any given angle; then, because of similar triangles, we know that those values will be the same when we see that same angle in any triangle, even if the hypotenuse of the triangle is not 1. For example, the unit circle tells us that the cosine of $50°$ is about 0.6428, so whenever we see a right triangle with a $50°$ angle in it, we know (because the triangles are similar) that the ratio of the adjacent leg to the hypotenuse will always be 0.6428, without having to measure the sides.
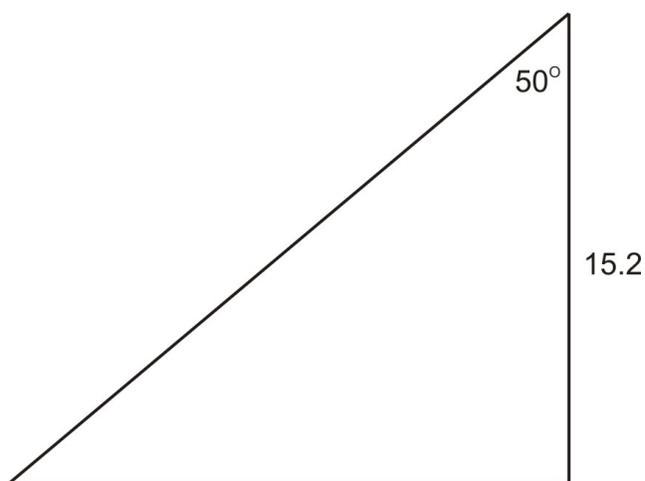


$$\frac{\text{adjacent}}{\text{hypotenuse}} = 0.6428$$
$$\text{adjacent} = \text{hypotenuse} * 0.6428$$
$$15.2 = \text{hypotenuse} * 0.6428$$
$$\text{hypotenuse} = \frac{15.2}{0.6428} \approx 23.6465$$

## Trigonometric Functions of Any Angle

**Reference Angles and Angles in the Unit Circle**

After example 1, you may need to show more explicitly how we know the ordered pair for 150° based on the ordered pair for 30°. Remind students that a 150° angle is the reflection across the $y-$axis of a 30° angle (refer to the earlier diagram), and remind them (and demonstrate visually) that when we reflect a point across the $y-$axis, its $y-$coordinate stays the same and its $x-$coordinate changes sign.

Example 2 provides another opportunity to make this clear. Each angle that has 60° as its reference angle is simply the angle we get if we reflect a 60° angle across the $x-$axis (putting it in the fourth quadrant), the $y-$axis (putting it in the second quadrant), or both (putting it in the third quadrant). If we reflect it across the $x-$axis, its $x-$ coordinate stays the same and its $y-$coordinate changes sign; if we reflect it across the $y-$axis, its $y-$coordinate stays the same and its $x-$coordinate changes sign; and if we do both, both coordinates change sign. So we can easily find the coordinates for any angle once we know its reference angle and which quadrant it is in.

### Trigonometric Function Values in Tables

When you arrive at the table of trig function values, you may want to encourage students to compare the values of the trig functions for pairs of supplementary angles (like 85° and 95°, or 125° and 55°). The table makes it clear that cosines of supplementary angles are equal, and sines and tangents of supplementary angles are opposites. Thinking in terms of reference angles will make it clearer why this happens: an angle between 90° and 180° has a reference angle that is equal to its supplement, so the values of the trig functions for that angle are closely related to the values for its supplementary angle.

Because of this fact, there is another way to find the answer to example 6a; challenge students to figure out what it is. (Hint: what is the reference angle of 130°?)

You may also want to encourage students to compare the sine and cosine values for pairs of complementary angles, like 35° and 55°; the table shows that the sine of an angle is equal to the cosine of its complement. This fact will be useful later, and the reason for it will be clearer when we study the unit circle in more detail.

### Points to Consider

Here's one way to explain the difference between the measure of an angle and its reference angle: when you start at the positive $x-$axis and rotate counterclockwise to get to the terminal side of the angle, the distance you've traveled is the angle measure. When you start at the terminal side of the angle and travel by the quickest route to the closest portion of the $x-$axis, the distance you've traveled is the reference angle. (Demonstrate this visually with at least one angle that is not in the first quadrant. For example, with a 240° angle, you can show that 240° is the clockwise distance from the positive $x-$axis, but 60° is the shortest distance to the closest part of the $x-$axis.)

An angle is the same as its reference angle only when it is between 0° and 90°.

The simplest way to answer question #2 is by considering how we find values of trig functions on the unit circle. The values of sine and cosine there are simply equal to the $y-$coordinate and $x-$coordinate, respectively, of the ordered pair that defines the angle, so the angles that have the same (or opposite) sine (or cosine) value will simply be the ones with the same (or opposite) $y-$coordinate (or $x-$coordinate).

### Review Questions

The function in problem 12 is fairly complex (although it can be simplified) and it shouldn't be immediately obvious what the graph will look like. Students should be able to simplify the expression under the square root sign based on their conjecture from the previous problem; after that, the best they can do is figure out what the function's values will be for a few key angles (30°, 45°, 60°, and so on), plot the points, and sketch a graph based on those points, and then graph the function on a calculator to compare it with their sketch.

## Relating Trigonometric Functions

### Reciprocal Identities

After explaining that we obtain the reciprocal of a fraction by flipping the fraction, you might need to clarify how to obtain the reciprocal of something that is not a fraction. For example, the reciprocal of $x$ is $\frac{1}{x}$, and that's because $x$ is equivalent to the fraction $\frac{x}{1}$, which we can flip to find the reciprocal. Similarly, a trig value like cos $\theta$ is equivalent to $\frac{\cos \theta}{1}$, so if sec $\theta$ is the reciprocal of cos $\theta$, that means it is equivalent to $\frac{1}{\cos \theta}$.

The identity $1 = \sin^2 x + \cos^2 x$ may not be immediately obvious; it hasn't previously been explicitly mentioned, although students were encouraged to discover it for themselves in Review Question 11 of the previous lesson. If you wish, you can demonstrate how to derive it from the definitions of sine and cosine and the Pythagorean Theorem: In right triangle ABC where c is the hypotenuse, by definition sin $A = \frac{a}{c}$ and cos $A = \frac{b}{c}$; therefore $\sin^2 A + \cos^2 A = \frac{a^2}{c^2} + \frac{b^2}{c^2}$. Then if $\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$, multiplying through by $c^2$ yields $a^2 + b^2 = c^2$, which we know is true because it is simply the Pythagorean Theorem.

### Domain, Range, and Signs of Functions

It's much easier to remember which trig functions are positive and negative in which quadrants if we simply note that in the first quadrant they are all positive; in the second quadrant only sin and its reciprocal, csc, are positive; in the third quadrant only tan and its reciprocal are positive; and in the fourth quadrant only cos and its reciprocal are positive. This can be summarized by the mnemonic "All Students Take Calculus": All positive, Sin positive, Tan positive, Cos positive.

(The second-best way to figure out which functions are positive where is to think about which functions depend on the $x-$coordinate, which ones depend on the $y-$coordinate, and which ones depend on both, and then figure out which coordinates are negative depending on which quadrant we are considering. For example, in the third quadrant, where both coordinates are negative, sin and cos will be negative (and therefore so will sec and csc), but tan (and therefore cot also) will be positive because $\frac{x}{y}$ is positive when both x and y are negative. But this method takes longer, and is more prone to error, than simply using the mnemonic above.)

### Points to Consider

It can often be easier to tell if an equation is *not* an identity: simply plug in a couple of values for the variables in the equation and see if the equation holds true. If it doesn't, the equation is not an identity; if it does, the equation might or might not be.

Similarly, you can verify the domain and range of a function by plugging in numbers that are inside or outside the domain or range and seeing what happens. If a number is in the domain of the function, it should yield a sensible result when you substitute it for $x$; if it's not in the domain, it should not yield any real-number result. If a number is in the range of the function, you should be able to get it as a result when you plug in some value of $x$; if it's not in the range, you shouldn't be able to for any value of $x$.

## Applications of Right Triangle Trigonometry

### Solving Right Triangles

If you are going through example 2 as a class, you may want to have students stop and think about how they might solve the triangle before walking them through the solution presented in the text. First, they should assess how much information they already have available—in this case, one side and two angles. (They may only notice the one angle measure that is written out in numerals—remind them that they also know the measure of the right angle.) Then they should think about what strategies they know for finding the other

sides and angles (Pythagorean theorem, trig ratios), and which numbers they would need to plug in for each of those strategies. Finally, considering what numbers they actually have available to plug in should give them some idea of which strategies they could effectively use.

To answer the question posed in the solution: Using the tangent to find the third side is better than using the sine because using the tangent allows us to plug in the side we were given at the beginning, whose length we know precisely, rather than the side we just found, whose length we only know approximately. It's always best to base our calculations on the most precise information we have available, so that rounding errors don't accumulate.

### Angles of Elevation and Depression

Clinometers are tools for measuring angles of elevation and depression; theodolites can measure horizontal angles as well. More information about both of these can be found on the Internet or in an encyclopedia.

For extra precision when measuring angles of elevation and depression, you should of course subtract several inches from your total height to estimate the distance from your eyes to the ground.

### Other Applications of Right Triangles

An explanation of how we know the information given in example 7: We know the distance between the moon and the earth based on calculations that will be explained later in the book. We know the angle between the moon and the sun at a given time because we can measure it directly from our vantage point on the earth. And finally, we know that the moon makes a right angle with the earth and sun at the first quarter (when the moon is halfway full) because exactly half of the portion of the moon that we can see is lit up by the sun, meaning that the sun must be shining exactly "sideways" on the moon.

### Points to Consider

In addition to the situations described in this lesson, we also might use right triangles to determine the shortest distance between two points on a grid (like a grid of city blocks), or to determine how long a ladder we need to reach a certain height on a building.

Any right triangle can be solved if we have enough information; at minimum, we need to know the length of at least two sides, or one side and one angle besides the right angle.

Trigonometry can solve problems at any scale because the trig ratios are the same for any size triangle as long as the angles are the same.

## 1.2  Circular Functions

## Radian Measure

### Radians, Degrees, and a Calculator

Most scientific and graphing calculators have a $\pi$ key, primarily to make calculating angles in radians easier. Make sure your students know where this key is on their calculators.

When a question like example 4 comes up, students may wonder how they are supposed to know that this angle measure is in radians and not degrees. After all, $\frac{3\pi}{4}$ is just a number like any other, so an angle of $\frac{3\pi}{4}$ degrees could exist too—which means we can't just assume any angle measure with   in it is in radians. And as we've seen in the text, measures like 1 radian and 2 radians are meaningful as well, so we can't assume any angle measure *without* $\pi$ in it is in degrees.

The solution to this conundrum is to assume all angle measures are in radians unless otherwise specified;

that's the convention used by mathematicians. So if you're asked what the cosine of 5 is, if the problem says just "5" and not "5°," assume it means 5 radians. But beware of typographical errors! If an angle measure doesn't include a degree sign, but is a suspiciously familiar round number like 60 or 90, it may be worth looking over the problem for signs that the author might have really meant to specify degrees and just left out the degree sign by mistake.

Example 7 contains a new concept: the inverse sine function. You may need to help students find the inverse sine on their calculators (usually they'll need to press the "$2^{nd}$" key followed by the "sin" key), and you may also need to explain the inverse sine function itself.

First, a brief review of inverse functions may be needed: remind students that the inverse of a function is simply what you get when you apply the function "backwards," so the input becomes the output and the output becomes the input. In the case of the sine function, normally the input is an angle measure (which can be any real number) and the output is a ratio of side lengths (which can be any real number between 1 and 1). The inverse sine function, therefore, takes a number between 1 and 1 as its input, and its output is the measure of an angle whose sine is that number. For example, the inverse sine of 1 is 90° (or $\frac{\pi}{2}$ radians), because the sine of 90° (or $\frac{\Pi}{2}$ radians) is 1.

The notation used here also bears explaining. Inverse functions are written with what looks like an exponent: the inverse of $f(x)$ is written as $f^{-1}(x)$, and the inverse of $\sin(x)$ is written as $\sin^{-1}(x)$. Emphasize that $\sin^{-1}(x)$ does not mean $\sin(x)$ raised to the power of $-1$, even though it looks as if it does. (Normally, as we learned in the previous chapter, placing the exponent right after the $f$ in $f(x)$ is the standard notation for raising a function to a power, and when raising a trig function like $\sin(x)$ to a power, we put the exponent right after the "sin" part. But when we want to raise $\sin(x)$ to the power of $-1$, we must write $(\sin\ x)^{-1}$ instead, so that we can use $\sin^{-1}(x)$ to designate the inverse sine of $x$.)

## Applications of Radian Measure

### Rotations

Example 1 contains a slight error: since the hour hand has rotated $\frac{1}{3}$ of the way from 11 to 12, the distance between the hour hand and the 12 is $\frac{2}{3}$, not $\frac{1}{3}$, of that twelfth of the circle; it is $\frac{\pi}{9}$ rather than $\frac{\pi}{18}$ radians.

### Length of Arc

The illustration for example 2 shows 12 feet as the diameter of the circle, but the solution worked out in the text is in fact based on the *radius* of the circle being 12 feet. If you wish, you can have students solve the problem both ways just for practice, but make sure to keep everyone on the same page about which formulation of the problem you are using at which time.

Perceptive students may try to solve example 3 the short way, by noticing that since the radius of the larger circle is $\frac{7}{4}$ that of the smaller circle, the circumference is also $\frac{7}{4}$ as great, and so a complete rotation of the smaller circle would be $\frac{4}{7}$ of a rotation of the larger circle. This is certainly a legitimate way of solving the problem, but you might encourage them to do it over again the way the book describes, just so they can get some practice doing calculations with arc lengths.

Additionally, there is a more precise way to express the answer to this problem. Instead of finding a decimal approximation for $\theta$ and then multiplying it by $\frac{180}{\pi}$ to approximate the angle measure in degrees, it is better to simply leave the value of $\theta$ in fraction form as $\frac{8\pi}{7}$, so that multiplying it by $\frac{180}{\pi}$ yields the answer $\frac{1440}{7}$ degrees.

### Area of a Sector

You may need to explain the setup of the equation for 1 radian. Basically, we are starting with the equation $2\pi$ (radians) $= \pi r^2$ (area) and dividing by $2\pi$ to make the left side equal 1 radian. Dividing the right side

by $2\pi$ then gives us $\frac{1}{2}r^2$ as the area.

The diagram given for example 4 contains a slight error ; you might encourage students to find it for themselves. (It's a tricky error to catch—just remember that $\frac{2\pi}{3}$ is not in fact $\frac{2}{3}$ of $2\pi$.) This is a good opportunity to remind them not to make the same error themselves, as it's quite a common one.

### Length of a Chord

Example 5 presents a slightly convoluted solving method, as it has students find *half* the length of the chord and then double it to get the final answer. In future, students will find it easier to simply use the formula for the *whole* chord length: twice the radius of the circle times the sine of half the angle, or $2r \, \sin\left(\frac{\theta}{2}\right)$.

Also, when applying the chord-length formula, it isn't actually necessary for the angle measure to be in radians (as it is with the arc-length and area formulas), because the sine of the angle is the same whether the angle is in radians or degrees. However, it is still useful to have students practice converting from degrees to radians.

## Circular Functions of Real Numbers

$y = \sin(x)$, **the Sine Graph**

Students should notice that the height of the point tracing out the sine graph at any given stage is exactly the same (in graph-units) as the height of the point moving around the circle. (This may not be immediately obvious because the sine graph and the circle graph are depicted on slightly different scales.)

$y = \cos(x)$, **the Cosine Graph**

The relationship between graph height and location on the circle is harder to see for the cosine graph, because the height of the graph represents the horizontal rather than the vertical location of the point on the circle. Imagining the circle rotated a quarter-turn to the left may help make the connection more visible, as you can now see the heights of the two points matching the same way they did on the sine graph—but your students probably needn't try this, as long as they understand the basic principle that the cosine represents the $x-$value of the corresponding angle.

$y = \tan(x)$, **the Tangent Graph**

It may seem strange that the tangent line can get infinitely long when the sine and cosine lines can't. Remind students that the length of the tangent line represents the ratio between the sine and cosine, and so it gets infinitely big as the cosine gets infinitesimally small.

To explain this in terms of the similar triangles shown in the diagram: The sides marked $t$ and 1 have the same ratio as the sides marked $y$ and $x$. As side $y$ gets longer, side $t$ gets longer—but as side $x$ gets shorter, side 1 can't get shorter because its length is fixed at 1. So if that side can't get shorter, side t has to get even longer to keep the ratio the same.

**The Three Reciprocal Functions:** $\cot(x), csc(x),$ **and** $\sec(x)$

You might stop and ask your students why it makes sense that 1 and $-1$ are the only values for which a function and its reciprocal are the same. (Hint: What has to be true of $y$ in order for $y = \frac{1}{y}$ to be true? Further hint: What happens when you solve that equation for $y$?)

The illustrations showing the cosecant segment for angles greater than $180°$ may be a little confusing, as the segment looks the same as it did for angles less than $180°$, but its length is now being described as a negative number. The reason it is now negative is that the segment is now pointing in the opposite direction from the line segment that forms the terminal side of the angle, whereas it was pointing in the same direction for angles less than $180°$.

To make it extra clear that the graph of the secant function is not made up of parabolas, you can point out that the secant graph has vertical asymptotes, whereas parabolas have no domain restrictions and extend infinitely far in both the positive and negative $x-$directions.

**Lesson Summary**

It's possible to express the domain restrictions on the cotangent and cosecant functions in a way that makes clearer their relationship to the domains of the tangent and secant functions. Instead of $\{x : x \neq n\pi$, where n is any integer$\}$, we can express the domain of the cotangent or cosecant as $\{x : x \neq n\left(\frac{\pi}{2}\right)$, where $n$ is any even integer$\}$. When we compare this to the domain of the secant or tangent, $\{x : x \neq n\left(\frac{\pi}{2}\right)$, where $n$ is any odd integer$\}$, we can see much more clearly that the cosecant and cotangent have the same pattern of asymptotes as the secant and tangent, just shifted by $\pi$ units.

## Linear and Angular Velocity

**Linear Velocity** $v = \frac{s}{t}$

You may need to stress that s represents distance and does not stand for "speed." Students will still be prone to forget this when plugging in values without thinking too hard, so they may need reminding when they slip up.

In example 2, students might need to be reminded that the bar over the 3 in $21.\bar{3}$ signifies a repeating decimal; the quantity being expressed is $21.233333333\ldots$ with the 3's continuing on forever.

**Angular Velocity** $\omega = \frac{\theta}{t}$

After finding the angular velocity in example 4, you may want to pause and explain why this answer makes sense: if the girls rotate through $\frac{1}{6}$ of the circle each second, it will take them 6 seconds to make a complete rotation, and 6 seconds is indeed the time we were given at the beginning of the problem.

Similarly, it is useful to check the linear velocities and see why they make sense. Since Lindsey is 7 feet from the center while Megan is 2.5 feet from the center, Lindsey should be traveling about $\frac{7}{2.5}$ or $\frac{14}{5}$ times as fast as Megan. Sure enough, their speeds are about 7.3 and 2.6 feet per second respectively, and $\frac{7.3}{2.6}$ is about the same as $\frac{14}{5}$.

Another problem for students to consider: if your arms are 2 feet long, and you swing a baseball bat that is also 2 feet long, how much faster is the end of the bat traveling than the end of your arms? (You can make up an angular velocity for the swing and have students find the two linear velocities, but you don't need to—it may be more educational to have them look at the problem more abstractly. The key insight here is that no matter what the actual speeds are, since the arc of the circle that the bat sweeps out has a radius twice as great as the arc of the circle swept out by the swinger's arms, the end of the bat will be traveling twice as fast. Students can see this by looking at the formula for linear velocity in terms of angular velocity, $v = r\omega$, and considering what happens if $r$ is doubled—or multiplied by any other constant.) What does this suggest about why we use baseball bats—or for that matter, tennis rackets, golf clubs, or croquet mallets?

You may also want to have students try solving the formulas backwards: for instance, if the girls' angular velocity in Example 4 were $\frac{\pi}{2}$ radians per second, what would be their linear velocity if the merry$-$go-round were still the same size?

## Graphing Sine and Cosine Functions

**Amplitude**

You may need to stress that the amplitude is the greatest distance the wave gets from the *center* of the

wave, so it is only half the distance between the minimum and maximum values.

### Period and Frequency

You may want to clarify the exact relationship between period and frequency, perhaps by helping students to work it out on their own. Remind them that an inverse relationship means that one quantity decreases when the other increases, and that it also means that the two quantities will yield a constant result when multiplied together. The three examples given in the text are a function with a period of $2\pi$ and a frequency of 1; a function with a period of $\frac{\pi}{4}$ and a frequency of 8; and a function with a period of $4\pi$ and a frequency of $\frac{1}{2}$. In each of these cases, what do we get when we multiply the period by the frequency? (Answer: $2\pi$.) So what does that suggest the relationship is between the two? (Answer: period $*$ frequency $= 2\pi$, which is more usefully expressed as either period $= \frac{2\pi}{\text{frequency}}$ or frequency $= \frac{2\pi}{\text{period}}$; both expressions are useful in different situations.)

$$y = csc(x)$$

Students may find it a little counterintuitive that the period of the cosecant graph is $2\pi$, because the graph divides up so neatly into $\pi-$sized chunks (and also perhaps because they've just seen that the period of the tangent graph is $\pi$ units). Stress that it takes two of those chunks, in this case, before the graph actually repeats itself, just as in Example 2 at the beginning of the lesson it took one "high" portion and one "low" portion of the graph together to make up one period.

### Transformations of Sine and Cosine Graphs: Dilations

It's a little hard to tell from the graphs of those linear functions that the line is being "stretched" vertically when the slope increases, because vertically stretching a straight line looks just like rotating it. You may want to also show graphs of the dilations of $x^2$ discussed in the text, because those graphs will make the stretching and shrinking a lot clearer.

Also, the statement "Constants greater than 1 will stretch the graph out vertically and those less than 1 will shrink it vertically" is really only part of the story. That is, it's true for positive constants, but negative constants will first flip the graph upside down and then stretch or shrink it vertically (stretch it if their absolute value is greater than 1, shrink it if it's less than 1.) And of course, a constant of 0 will shrink the graph all the way down to a straight horizontal line.

### Review Questions

Encourage students to sketch graphs for question 2. Also, remind them if necessary that an amplitude of $A$ units (where $A$ is the constant multiplier in front of the trig function) means that the graph goes both $A$ units above the $x-$axis and $A$ units below it, so the maximum is $A$ and the minimum is $-A$. (The exception, of course, is when $A$ is negative, as in part c; then $A$ is the minimum and $-A$ is the maximum.)

## Translating Sine and Cosine Functions

### Vertical Translations

Another way to find the answer to example 1 is to find the minimum and maximum of the base function, $\cos(x)$, and then subtract 6 from both of them.

### Horizontal Translations (Phase Shift)

Here is another way to explain the apparent "backwards-ness" of phase shift: when we look at a given number $x$, an expression like $\cos(x - 2)$ means "the cosine of the number that's 2 units to the left of this

one." So it's as though we're taking the value of the cosine function from 2 units to the left of "here," and moving it over to "here"—which means we're moving it 2 units to the right of where it started out. And of course we're doing the same thing with the whole function, so $\cos(x-2)$ describes the whole cosine function shifted 2 units to the right.

Yet another way to explain it is in terms of inverse functions. Students may remember from previous algebra classes that the graph of an inverse function is the graph of the original function reflected about the line $y = x$; in other words, flipped diagonally. This means that a horizontal shift of the original graph would result in a vertical shift of the inverse graph, and vice versa. Specifically, shifting the original graph to the right corresponds to shifting the inverse graph up, and shifting the original graph left is the same as shifting the inverse graph down.

Now, one way to find the equation for an inverse function is to swap the $x-$ and $y-$values of the original equation and then solve for $y$—so if the original equation was $y = 3x$, the inverse equation would be $x = 3y$, or in other words $y = \frac{x}{3}$. But note that if we add a constant to the $x-$value in the original equation, that constant ends up being subtracted from the inverse equation—if we start with $y = 3(x+4)$ instead of just $y = 3x$, the inverse is $x = 3(y+4)$, or $x = 3y + 12$, or $x{-}12 = 3y$, or $y = \frac{x}{3}{-}4$. Because everything gets reversed in an inverse operation, *increasing* the $x-$value of the original function means decreasing the $y-$value of the inverse function. That means the inverse function gets shifted down (not up), and that must mean the original function was shifted left (not right) when the $x-$value was increased. So, increasing the $x-$value means shifting the graph left, and vice versa.

One way to explain why sin and cos are just phase-shifted versions of each other is to recall that they are based on the $x-$ and $y-$coordinates of a point moving around the unit circle. These coordinates behave in the same way as the point rotates—they both oscillate between 1 and $-1$—it's just that they start out at different points in the cycle.

Another way to explain it is this: Recall that the sine of an angle is the cosine of the angle's complement (draw a triangle to see why this is so—the sine of one acute angle is the cosine of the other acute angle, and the two angles add up to 90° by the Triangle Sum Theorem). We can write this fact as $\sin(x) = \cos(90{-}x)$, and we can rewrite $\cos(90{-}x)$ as $\cos(-x+90)$. But the graph of $\cos(-x+90)$ is simply the graph of $\cos x$ flipped over vertically (since all the negative $x-$values become positive and vice versa) and shifted horizontally by 90° (or $\frac{\pi}{2}$ radians), and that happens to be the same as the graph we get if we shift it $\frac{\pi}{2}$ units in the other direction and *don't* flip it.

## General Sinusoidal Graphs

### Drawing Sketches/Identifying Transformations from the Equation

Some students may notice that translating a basic sine or cosine graph $\pi$ units horizontally is essentially the same as flipping it upside down (i.e. multiplying it by $-1$). This isn't a necessary fact to know, but it can be useful. For particularly curious students, here's how to explain why it is true:

When you add $\pi$ to an angle measure, you get another angle two quadrants away with the same reference angle. (Sketching a couple of angles will make this obvious.) That means all the trig functions for that new angle will be either the same or the opposite of the trig functions for the old angle. Now, in any two quadrants that are opposite each other, the sine function has opposite signs—it's positive in I but negative in III, and positive in II but negative in IV. Similarly, the cosine function is positive in I but negative in III, and negative in II but positive in IV. So, whenever we add $\pi$ units to an angle, the sine and cosine of the new angle are the negatives of the sine and cosine of the old angle—or, to put it more formally, for any angle $x$, $\sin(x + \pi)$ equals $\sin(x)$ and $\cos(x + \pi)$ equals $-\cos(x)$. In other words, applying a phase shift of $\pi$ units to the sine or cosine graph is the same as multiplying it by $-1$, or flipping the graph over.

(You may also notice that, since the sign of the tangent function alternates from quadrant to quadrant, the

tangent function keeps the same sign when you add $\pi$ to the angle. This is why the tangent function simply repeats itself after $\pi$ units, and a phase shift of $\pi$ units is equivalent to no change at all.)

Examples 1 and 2 demonstrate two different ways to approach the problem of sketching a graph: starting with the horizontal and vertical translations, or starting with the amplitude and frequency. Students will probably find they prefer one method or the other, and there's certainly no need to be strict about which one to use.

Also, some students may find it easier to sketch a complete curve at each step of the process until they end up with the final curve, while others may find it easier to simply sketch the key points of the graph, move them around as necessary, and not connect them with a curve until the final step. Again, either method should work fine; it may in fact be a good idea to point out that both methods exist.

### Writing the Equation from a Sketch

A second way to find the amplitude after finding the phase shift is just to subtract the phase shift from the maximum value of the function. For instance, in the example given, we would subtract 20 from 60 to get 40.

### Review Questions

Problems 6-10 should contain the instruction "Write an equation that describes the given graph." Of course, either sine or cosine may be used.

## 1.3  Trigonometric Identities

### Fundamental Identities

#### Reciprocal, Quotient, Pythagorean

This section reviews the definitions of the trig functions and the Trigonometric Pythagorean Theorem. The Pythagorean identities, you'll recall, were first covered in lesson 1.6, but here we see a slightly different way of deriving them. It may be useful to reinforce knowledge of the identities by walking through the derivations, but for many students this will simply be review.

You can point out that another way of expressing the reciprocal trig functions is as follows: $\csc \theta = \frac{c}{b}$; $\sec \theta = \frac{c}{a}$; $\cot \theta = \frac{a}{b}$. This way of expressing them is only useful for angles in triangles, though; for angles of rotation it may be more useful to think of them as $\frac{1}{\sin \theta}$ and so on.

#### Confirm Using Analytic Arguments

The diagram here with the vertical line representing a distance of t units may confuse students a little, but it simply demonstrates in a slightly unusual way the fact that any real number can correspond to a distance traveled around the unit circle, and therefore to an angle on that circle. Again, this should be review for most students.

There is a slight error in the last paragraph : where it reads "for points in the third and fourth quadrants we use angles formed by the radius that meets that point and the y axis," it should say "x axis." This is an opportunity to remind students about reference angles: the reference angle is always the angle made with the closest portion of the $x-$axis.

#### Confirm Using Technological Tools

Calculators generally do not have sec, csc, and cot keys, so instead one must use the cos, sin, and tan keys and then take the reciprocals of those functions using the reciprocal key (marked $x^{-1}$ or $\frac{1}{x}$). Stress once again that the keys marked $\sin^{-1}, \cos^{-1}$, and $\tan^{-1}$ will not yield the reciprocal functions $\csc, \sec$, and $\tan$,

but rather the arcsin, arccos, and arctan (that is, inverse sine, cosine, and tangent) functions.

### Alternative Forms

We see here that knowing just one trig function of an angle does not uniquely determine the angle, as there are always two quadrants it could be in (which two depends on whether the value of the trig function is positive or negative). Generally, knowing a second trig function of the angle, or at least knowing its sign, will narrow down which quadrant the angle could be in—but note that this won't be the case if the second function is just the reciprocal of the first function. So, for example, knowing the signs of the tangent and cotangent functions won't tell us what quadrant the angle is in, because the tangent and cotangent always have the same sign whatever the quadrant—and the same is true for the sine and cosecant, or the cosine and secant. But knowing the signs of, say, the tangent and secant functions will tell us which quadrant the angle is in, and the same is true for any two trig functions that are not each other's reciprocals.

## Verifying Identities

### Working with Trigonometric Identities

Students may get a little confused by the fact that the equations we're working with reduce to seemingly obvious identities. The point here, of course, is that we start with an equation that declares two complex expressions to be equivalent, and then prove the equation is true by showing it can be changed into a form that is much more obviously true.

Generally, as the text states, we only change one side of the equation at once. This is because we are not really changing the equation, merely changing how the equation is expressed. To take a simpler example: if we were asked to prove that $17 - (\frac{99}{3}) + 2(40 + 7) = 78$, we would not try to perform the same operations on both sides at once, because we don't need to; one side is already as simple as it can get. Instead, we would simplify the left-hand side, step by step, until we had shown that it does indeed equal 78, and so is the same as the right-hand side.

Here's one way to explain the difference between proving identities and solving equations: When we start out with a complicated equation like $17x + 5(x - 2) + 3 = 15$ and solve it for $x$, what we're really trying to do is answer the question "For what value(s) of $x$ is this equation true?" When we reduce the equation to $x = 1$, we've just shown that the original equation is true if $x = 1$, and false otherwise.

But if we tried to solve an equation like $3(x + 5) = 5x - 2(x + 5) + 25$, we would find that it reduces to $3x + 15 = 3x + 15$, which reduces even further to $0 = 0$, which is true no matter what $x$ is. So instead of proving the equation is true for certain values of $x$, we've discovered it's true for *all* values of $x$.

And that's exactly what we're doing in proving these trig identities: by showing that both sides of the equation reduce to the same expression, we're proving that the equation is always true for any value of . And once we know it's true, we can use it to make useful substitutions when solving trig problems in the future.

### Technology Note

Calculators can be useful in verifying identities, but it is dangerous to rely on them too much. If the graphs of two expressions look identical, it may mean the expressions are indeed equivalent, but it may also mean that the difference between them is just too small for the graph to show, or that they are only equivalent over this small interval. Since a graphing calculator can only show us part of a graph and can only draw it with limited precision, it cannot tell us for sure if two expressions really are mathematically equivalent.

What a graph can do, though, is tell us for sure if two expressions are *not* equivalent. If the graphs of the expressions look wildly different—in fact, if they look even a little bit different—then we can safely say that the expressions are not equal.

## Sum and Difference Identities for Cosine

### Difference and Sum Formulas for Cosine

Students may not immediately see why we are trying to find a formula for $\cos(a - b)$. The idea is that once we find such a formula, we can use it to find the cosine of an unfamiliar angle if we can express that angle as the difference of two angles we are familiar with. The sum formula will be similarly helpful.

Students who haven't encountered the distance formula for a while might need to be reminded of it. On the coordinate grid, the distance between two points $(a, b)$ and $(c, d)$ is equal to $\sqrt{(a-c)^2 + (b-d)^2}$; in other words, the square of the straight-line distance between the points is equal to the square of the horizontal distance plus the square of the vertical distance. This is derived directly from the Pythagorean Theorem; drawing a right triangle on the coordinate grid whose hypotenuse is the line between the two points will show how it is derived.

Note that two different identities are derived in this section. The diagrams and the table show how we derive the formula for $\cos(a - b)$; then the following lines show how we can use this formula to derive, in turn, the formula for $\cos(a + b)$. These identities may be easier for students to remember if expressed in words: "The cosine of the difference is the product of the cosines plus the product of the sines" and "The cosine of the sum is the product of the cosines minus the product of the sines."

### Use Cosine of Sum or Difference Identities to Verify Other Identities

Call attention to the labels "Identity A" and "Identity B" here. These labels aren't official names for these identities, but they will be used to refer back to them later in this lesson.

The identities themselves simply say that the sine of an angle is equal to the cosine of the angle's complement, and vice versa. We've already seen that this is true from working with angles in right triangles, as the sine of one acute angle in a triangle is equal to the cosine of the other, and the two angles are each other's complements.

### Use Cosine of Sum or Difference Identities to Find Exact Values

Here we see that the sum and difference formulas we have just learned can tell us the value of the cosine function for angles we haven't previously worked with, based on the values we already know for angles that are multiples of 30° or 45°. (Note, though, that this technique will still only tell us the cosines of angles that are multiples of 15°. That's because when you add or subtract angles that are multiples of 15° (which includes all angles that are multiples of 30° or 45°), the result is always also a multiple of 15°.)

### Technology Note

As before, note that calculators cannot tell us with absolute certainty if two expressions have identical values; they might, for example, be identical up to twenty decimal places, but differ after that point. However, if two expressions seem to have equal values when plugged into a calculator, there is a reasonably good chance that they are really equal—and more importantly, the calculator will clearly tell us if they are not equal at all. Hence, double-checking answers with a calculator is still a good idea.

## Sum and Difference Identities for Sine and Tangent

### Sum and Difference Identities for Sine

(For reference, Identities A and B are on page 244.)

The angle in example 1, $\frac{5\pi}{12}$, could of course be expressed in many different ways as the sum or difference of two other angles, but the way shown in the text is the most convenient way to express it in terms of angles we are already familiar with. Consider asking your students to explain why this choice is a good one. (You

might need to remind them, though, that $\frac{3\pi}{12}$ is the same as $\frac{\pi}{4}$ and $\frac{2\pi}{12}$ is the same as $\frac{\pi}{6}$ in order for them to see why these angles should be easy to work with.)

A shortcut to finding the cosines of the angles in example 2 is to notice that the side lengths are Pythagorean triples. That is, if the sine of angle $\alpha$ is $\frac{5}{13}$, the other leg of the relevant triangle must measure 12 (to complete the triple $5 - 12 - 13$) and so the cosine is $\frac{12}{13}$. Similar reasoning holds for angle $\beta$.

**Sum and Difference Identities for Tangent**

Once again, expressing the sum formula for tangent in words may make it easier for students to remember: "The tangent of the sum equals the sum of the tangents over 1 minus the product of the tangents." There isn't really a concise way to do this for the sine formulas, though.

Consider whether you want your students to memorize these and other trig identities or not. They definitely need to develop a good sense for when to use which ones, but the formulas themselves are the least important part of that knowledge; in fact, knowing how to derive the formulas may be more useful than simply knowing the formulas themselves. Instead of requiring that all the formulas be memorized for a test, for example, it might be more educational to supply a few of the formulas and require that students re-derive the others from the few given (after making sure, of course, that the formulas supplied are sufficient to derive the others from.)

Of course, some students might find that a little too challenging, but on the other hand, some will find it easier than memorization. Perhaps a good compromise would be to provide a few of the formulas so that students can derive the others if they want to, but not make that derivation a required part of the exam. Since students will presumably need to use any or all of the formulas to solve some of the exam questions, they will still have to either derive the formulas or have them memorized, but can do whichever of those two works best for them.

## Double-Angle Identities

### Deriving the Double-Angle Identities

There is really only one form of the double-angle identity for sine, but we see here that the double-angle identity for cosine can be expressed in several different ways. Students should be familiar with all of these formulas in case they encounter them "in the wild"; it's good to be able to recognize when an expression can be converted to $\cos 2a$, just as it's useful to be able to express $\cos 2a$ in terms of functions of a. Also, as will be seen later, some problems are easier or harder depending on which form of the double-angle identity we choose to work with.

### Applying the Double-Angle Identities

Just as in the previous lesson, noticing that 5 and 13 are part of a Pythagorean triple is a useful shortcut to finding $\cos a$.

### Finding Angle Values Given Double Angles

(Example 1 should read "$2x$ is a Quadrant II angle" (rather than "$x$ is a Quadrant II angle"), both in the introduction and in the second-to-last line of the table.)

The seventh line of the table may merit an explanation, as students whose algebra is rusty may wonder where the "$a$" comes from. Explain that it is simply a "dummy variable" that we choose to stand for $\sin^2 x$ so that the equation we are working with becomes a simple quadratic, which we know how to solve. Later, we will change $a$ back to $\sin^2 x$ in order to finish solving for $x$.

### Simplify Expressions Using Double-Angle Identities

In the example given here, you can show, if you wish, how choosing a different double-angle formula would make the expression we are working with more complicated, and hence why the one used here is the best one to use in this case.

For example, using $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ instead yields the following:

$$\frac{1 - \cos 2\theta}{\sin 2\theta}$$
$$= \frac{1 - (\cos^2 \theta - \sin^2 \theta)}{2 \sin \theta \cos \theta}$$
$$= \frac{1 - \cos^2 \theta + \sin^2 \theta}{2 \sin \theta \cos \theta}$$
$$= \frac{1}{2 \sin \theta \cos \theta} - \frac{\cos^2 \theta}{2 \sin \theta \cos \theta} + \frac{\sin^2 \theta}{2 \sin \theta \cos \theta}$$
$$= \frac{1}{2 \sin \theta \cos \theta} - \frac{\cos \theta}{2 \sin \theta} + \frac{\sin \theta}{2 \cos \theta}$$
$$= 2 \csc \theta \sec \theta - \frac{1}{2} \cot \theta + \frac{1}{2} \tan \theta$$

and at this point it isn't at all obvious how to show that this is equivalent to $\tan \theta$.

**Lesson Summary**

The first paragraph here describes a useful strategy for getting an idea of what sort of answers are reasonable before attempting to solve a trig problem: figure out, if you can, approximately where the given angles are and approximately what the values of the trig functions should be for those angles, or at least figure out upper or lower bounds on the trig functions based on whether the given angle is greater or less than an angle you are already familiar with, and whether the relevant trig functions are increasing or decreasing in the part of the unit circle where the angle is located.

This particular example contains a slight error , though: $\sin 45°$ is actually about 0.7, and $\sin 30°$ is 0.5, so the angle $\theta$ in the problem is actually between 30° and 45°, and $2\theta$ is therefore between 60° and 90°.

**Review Questions**

Some of these questions involve finding $\tan 2x$, which we haven't derived a formula for. In fact, there isn't a concise formula for the tangent of a double angle; however, once $\sin 2x$ and $\cos 2x$ have been found, $\tan 2x$ is simply the quotient of the two.

## Half-Angle Identities

### Deriving the Half-Angle Formulas

In the first and second derivations here, it's important for students to see that when we substitute a for $2\theta$, we can also substitute for $\theta$. This is how we are able to find an expression for the sine or cosine of $\frac{a}{2}$ based on that of $a$.

(In the tangent derivation, though, the text uses $a$ and $\theta$ interchangeably. This is because we don't have to make any actual half-angle substitutions there, but merely plug in the formulas for sine and cosine.)

Expressing the half-angle formula for cosine as $\sqrt{\frac{1+\cos a}{2}}$ instead of $\sqrt{\frac{\cos a + 1}{2}}$ may make it easier to remember, as it then looks more like the formula for sine.

### Use Half-Angle Identities to Find Exact Values

You might need to review how we know the sine of 225°. 225° is equal to 180° + 45°, which means it is a third quadrant angle with reference angle 45°, but you may want to draw it to make this clearer.

### Find Half-Angle Values Given Angles

For example 2, stress that although we have enough information to figure out the measure of angle $\theta$, we don't actually need to know it to apply the half-angle formula; the formula only requires us to know the cosine of $\theta$, which we already have.

You may want to explain, though, how we know that half of a fourth quadrant angle is a second quadrant angle. In general, halving a first or second quadrant angle will yield an angle in the first quadrant, and halving a third or fourth quadrant angle will yield an angle in the second quadrant. We can verify this numerically: half of an angle between 0° and 180° must be between 0° and 90°, and half of an angle between 180° and 360° must be between 90° and 180°.

### Using the Half- or Double-Angle Formulas to Verify Identities

Again, you may point out in the second line of this derivation that of the three possible expressions for cos $2\theta$, we've chosen the one that makes the numerator reduce to the simplest form.

### Technology Notes

Another way to demonstrate that $sin\frac{\theta}{2}$ does not equal $\frac{1}{2}\sin\theta$ is to pick a familiar pair of angles, such as 90° and 180°, and note that $\sin 90°$ is definitely not half of $\sin 180°$.

## Product-and-Sum, Sum-and-Product and Linear Combinations of Identities

### Transformations of Sums, Differences of Sines and Cosines, and Products of Sines and Cosines

Remind students not to mix up the formulas here with the sum and difference identities learned earlier: the formula for $\cos(\alpha + \beta)$ is not at all the same as the formula for $\cos\alpha + \cos\beta$!

For an extra challenge, you might ask students to derive the three formulas whose derivations are not shown, by applying similar reasoning to that used in the derivation that is shown.

### Transformations of Products of Sines and Cosines into Sums and Differences of Sines and Cosines

The product formulas shown here bear a certain resemblance to the sum formulas they are derived from, and students may be tempted to apply them after applying the sum formulas. For example, they may think, after they have determined that $\cos(a+b) = \cos a \cos b - \sin a \sin b$, that it would then be a good idea to plug in the expressions they've just learned for $\cos a \cos b$ and $\sin a \sin b$. But this will only give them a complex expression in terms of $\cos(a + b)$ and $\cos(a - b)$, which won't help them much since the value of $\cos(a + b)$ is what they were looking for to begin with.

The key thing they need to understand is that each of these identities is a tool to be used in different situations, depending on what knowledge they already have. If they need to find the sine or cosine of an angle, and that angle can be expressed as a sum or difference of two angles of which they already know the sine and cosine, then the sum formula from earlier is useful. If they need to know the product of two sines or cosines, and they don't know the sines or cosines themselves, but do know the sine and cosine of the sum and difference of those two angles, then the product formula learned here is useful. In general, they should form the habit of writing down exactly what it is they are looking for and then considering which tools in their possession might apply to that particular situation.

### Linear Combinations

You may need to define the term "linear combination" for students who haven't encountered it before. A linear combination of two quantities is simply a multiple of one quantity plus a multiple of the other—so, for example, a linear combination of $\sin x$ and $\cos x$ would be any expression of the form $a \sin x + b \cos x$, where $a$ and $b$ are any two numbers (real numbers, unless otherwise specified). Students will use linear combinations in their study of vectors in the next chapter.

## 1.4 Inverse Functions and Trigonometric Equations

### Inverse Trigonometric Functions

**Inverse Functions**

The definition of "one-to-one" bears reviewing here; a function is one-to-one if, in addition to having at most one $y-$value for every $x-$value, it also has at most one $x-$value for every $y-$value.

Students may get confused about the "inverse reflection principle." It doesn't mean that the graph of $f(x)$ or the graph of $f^{-1}(x)$ is symmetric about the line $y = x$, even though in this case both graphs look as though they almost are. Instead, it means that the graph of $f^{-1}(x)$ is what you get when you flip the graph of $f(x)$ about that line. Comparing the two graphs shown in the text should make this clearer.

The inverse reflection principle, in turn, should make it clear why a function has to pass the horizontal line test in order for its inverse to also be a function. When the graph of a relation is flipped about that diagonal line to create the graph of the inverse relation, any horizontal lines drawn through the graph of the original relation would become vertical lines drawn through the graph of the inverse relation—so if the original relation would not pass the horizontal line test, it follows that the inverse relation would not pass the vertical line test, and so would not be a function.

When students work through the examples in the text, they should make sure their calculators are in degree mode. With a calculator, it is much easier to find inverse trig functions in degrees than in radians, because a calculator in radian mode will generally only give decimal approximations for the answers instead of telling you precisely what the answers are in multiples of $\pi$. (When a calculator tells you that the inverse cosine of a number is 1.308996938, it's not easy to guess that that's the same as $\frac{5\pi}{12}$.) Some models of calculator will give answers in multiples of $\pi$, so students who are using those models may not have any difficulty with radian mode, but they should still generally stick with degree mode in order to be on the same page with everyone else.

**Points to Consider**

The inverse relations of the trig functions are one-to-one, but they are not functions. Under the restricted domains that will be discussed later, they are one-to-one functions.

**Using the "Inverse" Notation**

The graphs of $\sin(x)$ and $\sin^{-1}(x)$, as shown, do not look quite like they are each other's reflections about the line $y = x$, but that is simply because they are drawn on different scales. Stretching the graph of $\sin^{-1}(x)$ in the $y-$direction and shrinking it in the $x-$direction would make its resemblance to the graph of $\sin(x)$ more obvious. The same holds for the inverse cosine and tangent graphs.

**Points to Consider**

We can determine exact values for inverse trig functions when those functions correspond with the various special angles on the unit circle.

# Exact Values of Inverse Functions

### *Ranges of Inverse Circular Functions*

### Domain and Range of the Circular Functions and their Inverses

Remind students if necessary that the notation $|q|$ stands for the absolute value of $q$, which is the positive difference between $q$ and zero—in other words, $q$ if $q$ is positive, and $-q$ if $q$ is negative (so $|q|$ is always positive).

### *Exact Values of Special Inverse Circular Functions*

### Introduction

A much easier way to do the problems in example 1 is to think of cot, csc, and sec as being $\frac{adj}{opp}, \frac{hyp}{opp}$, and $\frac{hyp}{adj}$ respectively (we can derive these easily from the definitions of tan, sin, and cos), so that we don't have to find the reciprocals later by dividing fractions.

### Points to Consider

The inverse composition rule has not previously been discussed in this book, although students may have encountered it in other courses. It states that if the functions $f(x)$ and $g(x)$ are inverses of each other, then $f(g(x))$ and $g(f(x))$ both simply equal $x$. In other words, if we take an $x-$value, apply a function to it, and then apply that function's inverse to the result, we should get the same $x-$value back again. This makes sense since the inverse of a function is really just the function itself applied "backwards."

For example, the inverse of $f(x) = 3x$ is $g(x) = \frac{x}{3}$. According to the inverse composition rule, therefore, both $3\left(\frac{x}{3}\right)$ (that is, $f\left(\frac{x}{3}\right)$) and (that is, $g(3x)$) should equal $x$— and indeed they do.

So, does the rule apply to trig functions? Yes, if we use the appropriate domain restrictions. For example, the tangent of $45°$ is 1, and the inverse tangent of 1 is $45°$, so $\tan(\tan^{-1}(1)) = \tan(45°) = 1$, and $\tan^{-1}(\tan(45°)) = \tan^1(1) = 45°$. But the rule doesn't apply if we pick an angle measure outside of our restricted domain. The tangent of $225°$ is also 1, but the inverse tangent of 1 is still $45°$. (Of the many angles whose tangent is 1, $45°$ is the only "official" inverse tangent of 1. Similarly, although 2 and $-2$, when squared, each yield 4, 2 is the only "official" square root of 4.) So $\tan^{-1}(\tan(225°))$ would equal $45°$ instead of $225°$.

(This rule will be explored in more detail in the next section.)

# Properties of Inverse Circular Functions

### *Derive Properties of Other Five Inverse Circular Functions in terms of Arctan*

### Composing Trigonometric Functions with Arctan

The notation in this section may be a bit confusing; students may wonder why we would want to find the values of expressions like $\sin(\tan^{-1}(x))$ and $\csc(\tan^{-1}(x))$, or may be unclear on what these expressions really mean. Basically, all we are doing here is the same thing we did in chapter 1, when we were given, say, the sine of an angle and had to find the cosine or the tangent of that same angle without knowing the measure of the angle itself. Based on our knowledge of right triangles and the Pythagorean Theorem, we know that if the tangent of an angle is $x$, we can find all the other trig ratios in terms of $x$ just by drawing an appropriate triangle. If one leg of the triangle measures $x$ units and the other measures 1 unit (making the tangent $\frac{x}{1}$, or just $x$), then we can find the length of the hypotenuse, and once we know the lengths of all three sides, all the trig functions are simply ratios of certain pairs of sides.

So, for example, part b of Example 1 simply asks "What is the sine of an angle whose tangent is 1?" and the

answer can be found by drawing a triangle whose legs both measure 1, finding the length of the hypotenuse, and then finding the sine of the acute angle.

**Points to Consider**

It is indeed possible to graph these composite expressions, as they are simply algebraic functions. Analyzing them will show that their domains are unlimited but their ranges are limited, because the expression $(x^2 + 1)$ can only take on values greater than or equal to 1.

### *Derive Inverse Cofunction Properties*

**Cofunction Identities**

Technically, the graph of $y = \sin\theta$ is not really the graph of $y = \cos\theta$ shifted $\frac{\pi}{2}$ units to the right, but rather the graph of $y = \cos\theta$ flipped upside down and shifted $\frac{\pi}{2}$ units to the left. Since the resulting graph is exactly the same, though, you needn't stress this technicality, but it's useful to keep in mind in case some sharp student spots it. (You might, in that case, challenge them to work out why those two different operations are equivalent.)

### *Find Exact Values of Functions of Inverse Functions using Pythagorean Triples.*

The phrase "functions of inverse functions" may be confusing. In this case it simply means "trig functions applied to inverse trig functions," or in other words "inverse trig functions plugged into trig functions"—for example, $\cos(\sin^{-1}(x))$ or $\tan(\cos^{-1}(x))$—which is just the sort of thing we were working with earlier in the lesson.


## Applications of Inverse Circular Functions

### *Revisiting $y = c + a\cos b(x - d)$*

**Transformations of $y = \cos x$**

You may want to walk through examples 1 and 2 in more detail, as these concepts haven't been covered since chapter 2.

For example 1, first we need to pull out the amplitude, frequency, phase shift, and vertical shift from the equation: they are $3, 2, \frac{-\pi}{4}$, and 5 respectively. Then we need to think about how all of these values affect the basic cosine graph: it will be stretched vertically by a factor of 3, compressed horizontally by a factor of 2, shifted $\frac{\pi}{2}$ units to the left, and shifted 5 units up. Only after determining all this can we actually draw the graph.

For example 2, we perform almost the same procedure in reverse. First we inspect the graph to discover the amplitude, frequency, phase shift, and vertical shift: they are $5, 4$ (because the period is $90°$), $-20°$, and $-3$, respectively. Then we recall where those numbers fit into the equation, and then we can write the equation. (The equation given in the text should say $20°$ where it says $70°$.)

**Points to Consider**

Given an equation for $y$ in terms of $x$, it is generally possible to solve for $x$ in terms of $y$ (that is, to find the inverse of the equation); the result simply may not be a function. In this case, since we know we can find the inverse of the plain old cosine function, it seems reasonable that we can still find an inverse of the cosine function when it is stretched and shifted as it is here. In the next part of this lesson, we will find out how.

### *Solving for Particular Values in Trigonometric Equations*

**Points to Consider**

Degrees and radians are simply two different ways of expressing the same angle measures, so anything that can be done when working in degrees can also be done when working in radians.

### *Applications, Technological Tools*

### Examples

In Example 1, 4.34 seconds is one time when the dolphin is at a height of 4 feet, but it isn't the first time or even the only time. Try having students also determine the first time the dolphin reaches that height, and two other times when it does so again.

(They may think that since the graph has a period of 3 seconds, the dolphin reaches a height of 4 feet every 3 seconds. This is only partly true; the dolphin does reach that height at 4.34 seconds and every 3 seconds thereafter, but it also reaches that height at 3.66 seconds and every 3 seconds thereafter. Any height on the graph that is not a maximum or minimum will be reached not once, but twice per period: once on the way up and once on the way down. Tracing the graph should confirm this.)

## Trigonometric Equations

### *Solving Trigonometric Equations Analytically*

### Introduction

Before beginning this lesson, you will probably need to review all of the trig identities that were introduced in chapter 3: the Pythagorean identities, sum and difference identities, double- and half-angle identities, and sum and product identities. (If you haven't covered that chapter, you'll need to introduce those identities some other way, as they will be needed to solve the problems in this lesson—except for the double- and half-angle identities, which are covered in the next lesson.)

Students may try applying trigonometric identities to example 2. That isn't a useful technique, though, because the only trigonometric expression in the equation is already as simple as it can get. All that remains is to perform basic algebra to reduce the left-hand side of the equation to $\tan x$, and then to take the inverse tangent of both sides to get $x$ by itself. Except for the inverse tangent step, this problem has more to do with algebra than with trigonometry (proving that algebra skills are essential even as we study more advanced math!).

(Without a calculator, students may also forget the inverse tangent of $\sqrt{3}$. Remind them if necessary to draw an appropriate right triangle to figure it out.)

### Points to Consider

Any equation-solving method that works for non-trigonometric equations should also work for trigonometric ones, if applied appropriately. And it is certainly possible that a quadratic equation with trigonometric expressions in it might turn up—for example, $\sin^2 x + \sin x + 2 = 0$. We would first solve this equation for $\sin x$, using any standard technique for solving quadratics, and then we would take the arcsine of both sides to find $x$.

### *Solve Trig Equations (Factoring)*

### Introduction

"Principal values," in case students are confused by this, is simply another term for the values that are in the limited domain of a trig function, or in other words are in the range of the inverse function. In example 1, when $\sin x = \frac{1}{2}$, the principal value of $x$ is $\frac{\pi}{6}$ or $30°$, because that is the one $x-$value between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{1}{2}$.

### Points to Consider

The quadratic formula will work on trig equations; we just need to remember that we are solving for $\sin x$ rather than for $x$, and still need to take the arcsine afterward.

## Trigonometric Equations with Multiple Angles

### *Solve equations (with double angles)*

### Double Angle Identity for the Sine Function

Example 1, in addition to the double angle identity, also employs algebraic techniques that students may not have encountered for a while, such as the zero product property. It might be necessary to go through the last few steps slowly.

### Double Angle Identity for the Cosine Function

There is a typo in the first formula derived here: the final line of the derivation should have a minus sign in place of the second equals sign. Also, there should be a line break after the line in the next paragraph that reads $\cos 2a = 2\cos^2 -1$; the next three lines are a separate derivation of a different formula. (All three formulas are summarized immediately below.)

To solve example 2, we must note that $4\theta$ is twice $2\theta$, so the double-angle formula applies if we plug in $2\theta$ in place of the usual $\theta$.

### Double Angle Identity for the Tangent Function

You may want to point out that the solution to example 4 makes a great deal of sense if you think of angles as rotations around the unit circle. Remember that $\sin x$ is equal to the $y-$coordinate of the point on the unit circle that corresponds to the given angle $x$. (Using $x$ to stand for the measure of the angle can get confusing—remember that it does stand for the angle measure, in this case, and not for the corresponding $x-$coordinate.) The question this problem asks, therefore, is "What angle has the same $y-$coordinate as another angle that's twice as big?" Now, two angles can only have the same $y-$coordinate if they are the same angle, or if one of them is the reflection of the other across the $y-$axis. And the only angle whose reflection across the $y-$axis is also twice as big as itself is an angle of $\pi$ (or $-\pi$) radians (a rotation of half a circle). $\pi$ is outside the range of permissible answers we were given, so $-\pi$ is the solution we want. 0 is also a solution, because 0 doubled is still 0 and so an angle of measure 0 has the same coordinates as an angle twice as big.

### *Solving Trigonometric Equations Using Half Angle Formulas*

### Half-Angle Identity for the Sine Function

Basically, what we are doing here is applying the double-angle formula more or less in reverse to find the half-angle formula. Because we have to take the square root of both sides, however, we end up with two possible answers, one positive and one negative. We can tell which one is correct in any particular case, though, based on what quadrant the angle $\frac{\alpha}{2}$ is in: if it is in the first two quadrants, its sine must be positive, and if it is in the third or fourth quadrant, its sine must be negative.

(Similar reasoning is used below for the cosine half-angle formula.)

Example 1 should read "Use the half angle formula for the sine function to determine the value of $\sin 15°$" rather than $\sin 30°$.

## Equations with Inverse Circular Functions

### *Solving Trigonometric Equations Using Inverse Notation*

**Introduction**

Students may be prone to mix up the ranges for the various inverse functions, and they may be baffled as to why the ranges are seemingly arbitrarily different. The rationale for choosing those particular ranges is that each of them represents two adjacent quadrants, in one of which the given function is positive and in the other of which it is negative. That way, the whole set of possible values for the trig function is covered, but none of the values are repeated.

For the cosine function, for example, it is convenient to pick the first two quadrants, because the graph of the cosine function over the interval from 0 to $\pi$ hits every possible $y-$value exactly once. (The same holds true for the secant function, as it is the reciprocal of cosine.) But for the sine function, that interval won't work; the graph of the sine function from 0 to $\pi$ hits all the positive $y-$values twice, but none of the negative values. Instead, we use the interval from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, or the fourth and first quadrants, because over that interval the sine function hits all the possible $y-$values once each. (We could also use the interval from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, or the second and third quadrants, but it's more convenient to use an interval that includes 0.) And of course we use that same interval for cosecant, the reciprocal of sine. (For the cosecant and secant functions, though, we have to leave out the point in the very center of the range where the sine or cosine equals 0 and the cosecant or secant is therefore undefined.)

Finally, for the tangent and cotangent functions, we choose an interval that includes all possible values of the function while avoiding the function's asymptotes; that is why the endpoints are excluded from the ranges of those inverse functions, while the other functions' ranges include their endpoints.

Note: In the graphs in this section, the degree signs should actually be $-1$'s; for example, "$y = sin°x$" should read "$y = sin^{-1}x$."

**Points to Consider**

We haven't yet covered any identities that will help us express inverse trig functions in other ways. However, the identities we've learned so far can help us when we encounter expressions that have both trig functions and inverse trig functions in them, as we can use them to simplify the non-inverse functions.

***Solving Trigonometric Equations Using Inverse Functions***

**Points to Consider**

This isn't a trick question; we can indeed use identities to solve trig equations, though we won't always be able to in any particular case.

***Solving Inverse Equations Using Trigonometric Identities***

**Points to Consider**

We will see the applicability to real-world problems in the following review questions. You might encourage students to think of some more cases where these techniques could be applied.

# 1.5 Triangles and Vectors

## The Law of Cosines

**Introduction**

The definition of "oblique" (non-right) may be worth stressing briefly, as students may otherwise confuse it with "obtuse." (Adding to the confusion is the fact that obtuse triangles are also oblique triangles, although not all oblique triangles are obtuse.)

**Derive the Law of Cosines**

The fourth line of the derivation here contains a clever trick that is worth explaining. The text explains why we can substitute $a \cos C$ for $x$, but it may not be obvious why it is a good idea to do so. The reason is that it lets us express $c^2$ solely in terms of $a, b$, and $C$, so we now have a formula we can use for any triangle without needing to draw an altitude like BD again.

**Side of an Oblique Triangle (given the other two sides)**

"Note that the negative answer is thrown out as having no geometric meaning in this case" may bear explaining. The final step of example 1 involves taking the square root of both sides, which yields two possible answers, positive and negative. But we can safely disregard the negative answer because the number we are looking for represents the length of a line segment, which must be positive. This will be the case whenever we use the Law of Cosines.

Part 2 of the Real-World Application problem involves using the Law of Cosines differently, to find an angle when the sides are known instead of a side when the other two sides and an angle are known. This technique won't actually be explained until the next section, so you may need to walk through this example very carefully, or just skip ahead to the next section and then come back to it.

**Identify Accurate Drawings of General Triangles**

A problem like Example 3 could of course be done faster if we were in a hurry; we could simply skip to part 2, since finding out what the angle should be would also tell us if the angle given was the correct one.

**Points to Consider**

1) If we apply the Law of Cosines to a right triangle, the term $2ab \cos C$ becomes zero because $\cos 90°$ is zero. The Law of Cosines then reduces to the Pythagorean Theorem.

2) It is possible to solve the triangle completely through repeated applications of the Law of Cosines. Knowing two sides and the included angle lets us find the third side; then we can apply the theorem backwards, plugging in the three sides to find either of the missing angles; and then we can do this again to find the other missing angle, or simply find it by the Triangle Sum Theorem.

3) We cannot use the Law of Cosines if we only know one or no side lengths; if we know two side lengths but no angles; or if we know two side lengths and one angle, but the angle is not between the two sides. (It may seem that we could still apply the Law of Cosines in the latter case, but for some triangles it turns out that that yields two positive answers, and there is no way to tell which is correct. This is the Ambiguous Case described in a later lesson.)

4) Students may need to experiment a bit to answer this question. If they are stumped, remind them of the Triangle Inequality they may have learned in Geometry: the sum of any two sides of a triangle must be greater than the third side. Any set of three numbers such that one of them is greater than the sum of the other two is a set of numbers that do not form a triangle.

## Area of a Triangle

**Derive Area $= \frac{1}{2}bc \sin A$**

The second line of the derivation here employs much the same trick as was used to derive the Law of Cosines in the previous lesson, and for similar reasons.

In estimating the cost of the pool cover here, we are simply calculating the exact cost by multiplying the precise area in square feet by \$35, for simplicity's sake. For a small extra challenge, ask students to find the total cost if the price is a more realistic \$35 per square foot or fraction thereof. (To solve this, they need to

round up the area to the nearest whole number and then multiply by $35.) For a bigger extra challenge, ask them to find the cost if the cover of the pool needs to be 1 foot longer on each side than the pool itself. (To solve this, they would use the same formula as before, but $b$ and $c$ would be 25 and 27 instead of 24 and 26.)

### Find the Area Using Three Sides–SSS (side-side-side) Heron's Formula

Observant students might worry that the terms (s-a), (s-b), and (s-c) in Heron's Formula could end up being negative, which could make the whole expression under the square root sign negative and make it impossible to find the square root. However, inspecting the formula for s shows us that the term (s-a) can only be negative if a is greater than b+c (similar reasoning holds for (s-b) and (s-c)), and the Triangle Inequality (mentioned in the notes to the previous lesson) guarantees that this can never be the case.

Incidentally, calculators will be needed throughout this chapter, as we are no longer working with special angles whose trig ratios we know, or even angles we can look up in tables like the one in chapter 1.

### Applications, Technological Tools

The sailboat diagram may be somewhat confusing. The jib sail is the gray shaded area on the left-hand side of the mast, and the line labeled "$x$" is the rope attaching the sail to the mast.

### Points to Consider

1) The Triangle Inequality is the answer to this question as well. If any one side were greater than half the perimeter, it would be greater than the sum of the other two sides, which the Triangle Inequality tells us is impossible.

2) In these three cases, we actually don't have enough information to solve the triangle or find its area using the techniques covered so far. We need to use the Law of Sines, which is the subject of the next lesson.

3) Applying Heron's Formula in reverse, though tedious, will yield the third side if the first two sides and the area are known.

## The Law of Sines

### Introduction

Note that in the diagram accompanying the Real-World Application problem, A represents Chicago and $C$ represents Buffalo; make sure students don't get caught thinking $B$ is Buffalo and $C$ is Chicago.

The phrase "the side is not included" may also cause confusion. "Included" is being used here in the technical sense, so "not included" means that the side is not in between the two angles—even though the length of the side is clearly "included" in the set of information we are given.

### Derive Two Forms of the Law of Sines

An interesting exercise is to apply the Law of Sines to a right triangle to verify its accuracy. If $a$ and $b$ are the legs of the triangle and $c$ is the hypotenuse, then $\frac{a}{\sin A} = \frac{a}{\frac{a}{c}} = c$. Similarly, $\frac{b}{\sin B} = \frac{b}{\frac{b}{c}} = c$, and $\frac{c}{\sin C} = \frac{c}{1} = c$ (because $\sin 90° = 1$). As predicted, the ratios of all three sides to the sines of their opposite angles are equal.

### AAS (Angle-Angle-Side)

An isosceles triangle appears in the Real-World Application problem here; we know that sides $BC$ and $DC$ are congruent because the angles across from them are congruent. A question for students: how could the Law of Sines have told us this if we didn't already know it? (Hint: if $\frac{a}{\sin A} = \frac{b}{\sin B}$, and $A = B$ (so $\sin A = \sin B$), what does that tell us about $a$ and $b$?)

### ASA (Angle-Side-Angle)

The ASA case is very close to the AAS case; in fact, by using the Triangle Sum Theorem to find the third angle, we are really turning the ASA case into the AAS case, because we now know two angles and a side that is not between them.

For students who are developing headaches trying to memorize all these three-letter acronyms, there is a simpler way to figure out when it is possible to use the Law of Sines. If you know two angles of a triangle, you really know all three angles, so if you know any two angles and one side, you can use the Law of Sines to find the other two sides.

### Applications

The diagram in situation 2 is incompletely labeled : the top of the mountain should be marked $M$, the right angle $N$, and the $127°$ angle $U$. Side $u$ is then the same as side $x$.

### Points to Consider

1) We still can't use the Law of Sines or Cosines if we don't know any of a triangle's side lengths, or if we only know one side and one angle. Also, we can't use them if we only know two sides and an angle that is not between them.

2) Two angles can have the same sine if they are each other's complements. This is why we can use the Law of Sines to solve for a side, but not for an angle; if we used it to solve for an angle, we might get two possible angles and not know which was correct.

3) With the Laws of Sines and Cosines together, we can solve any triangle if we know all three sides; two angles and one side; or two sides and the angle between them.

## The Ambiguous Case

### Introduction

Note that the cases where the Law of Sines is useful are the cases where we know at least two angles. Thanks to the Triangle Sum Theorem, knowing two angles of a triangle really means we know all three angles, so the Law of Sines is useful when we know all the angles of a triangle (plus at least one side) and just need to find the remaining sides. It may therefore be useful to think "Use the Law of *Sines* to find the *sides*." (Just remember that this little saying doesn't always apply when we only know one angle—sometimes we need the Law of Cosines in that case.)

### Possible Triangles Given SSA

In the case where $a < b$, we can see why comparing $a$ with $b \sin A$ tells us how many solutions there are if we think in terms of the Law of Sines. The Law of Sines tells us that $b \sin A = a \sin B$, and that equation in turn tells us the following things:

1) If $a$ is **less than** $b \sin A$, $\sin B$ must be **greater than 1** in order to make $a \sin B$ equal to $b \sin A$. (If $a$ by itself isn't "big enough" to equal $b \sin A$, it has to be multiplied by something greater than 1 to make the whole expression big enough.) But there is no angle whose sine is greater than 1, so there is no solution.

2) Conversely, if $a$ is **greater than** $b \sin A$, $\sin B$ must be **less than** 1 in order to make $a \sin B$ equal to $b \sin A$. But for any given value of the sine function between 0 and 1, there are two different angles between $0°$ and $180°$ that correspond to it—one acute angle and one obtuse angle, which are each other's supplements. So there are two possible solutions.

3) Finally, if $a$ is **equal to** $b \sin a$, $\sin B$ must equal 1 to keep the equation true—and there is only one angle between $0°$ and $180°$ whose sine is 1: a right angle. So there is one solution.

**Points to Consider**

1) One way to write the Law of Sines is $\frac{a}{b} = \frac{\sin A}{\sin B}$. This tells us that if a is greater than $b$, $\sin A$ is also greater than $\sin B$. The sine function is greater the closer an angle is to $90°$, so A must be closer to $90°$ than $B$.

- **If $A$ and $B$ are both less than $90°$**, this means that $A$ is greater than $B$.

- **If $A$ is $90°$ or greater**, we already know $A$ is greater than $B$ because only one angle in a triangle can be $90°$ or greater.

- **If $B$ is $90°$ or greater**, then $A$ being closer to $90°$ than $B$ means that $(90°–A)$ must be less than $(B–90°)$, which means $A + B > 180°$—but that's not possible if $A$ and $B$ are two angles of a triangle, so that means $B$ just can't be $90°$ or greater.

So, the Law of Sines tells us that if $a > b$, $A > B$ and (more importantly) $B < 90°$. This means that if we apply the Law of Sines when $a > b$ and get two possible values for angle $B$, since one will be acute and one will be obtuse, we know that only the acute one can be correct.

(Applied more generally, the Law of Sines tells us that the biggest angle is opposite the biggest side and the smallest angle is opposite the smallest side. This is often useful to know.)

2) This was covered in the section above.

3) We can check whether each of the possible angles makes the total angle sum greater than $180°$. If it does, we throw it out; if not, it is a correct solution.

## General Solutions of Triangles

### Summary of Triangle Techniques

Students should note that by applying the techniques in this table repeatedly, they can find *all* of the missing sides or angles in a triangle as long as they start out with enough information to find one of them.

### Using the Law of Sines

In the Real-World Application problem, students are told to find the biggest angle first using the Law of Cosines in order to avoid dealing with the Ambiguous Case later when using the Law of Sines. This is because, as discussed in the notes to the previous lesson, only the biggest angle in a triangle can be obtuse, so finding the biggest angle first ensures that the other angles will be acute and will thus have only one possible value.

However, students should beware inaccurate diagrams! In a drawing that is not to scale (or one that *might* not be to scale), the angle that looks the biggest may not be the biggest. Instead of looking for what seems to be the biggest angle in the diagram given, if we know the exact side lengths we should make use of the fact that the biggest angle is opposite the longest side. (We can derive this fact from the Law of Sines, as described earlier.)

It may be best to skip the part that describes converting the angles to headings, as those headings can't be found accurately without knowing which way is due north .

### Points to Consider

1) It's possible, but never really necessary, to use the Law of Sines before the Law of Cosines. There are times when we have enough information to use the Law of Sines but not the Law of Cosines, but in those cases, applying the Law of Sines always gives us enough information to finish solving the triangle without needing the Law of Cosines any more. Specifically, after applying the Law of Sines, no matter what information we

started out with, we always end up knowing at least two sides and two angles, which means we really know two sides and three angles (by the Triangle Sum Theorem)—so one more application of the Law of Sines will give us the third side, and that's all we have left to find.

2) If we know three sides and one angle, we could apply the Law of Sines, but we might then find ourselves in the Ambiguous Case. To avoid that, we might prefer to use the Law of Cosines. In any other case where both laws are applicable, though, the Law of Sines is generally preferable simply because it is easier.

3) In both cases where the Law of Cosines is applicable, we end up knowing all three sides and one angle after we apply it. Therefore, we can then switch to the Law of Sines (SSA case), although as just mentioned, we may prefer to stick with the Law of Cosines instead.

## Vectors

### Introduction

In the definitions of displacement, velocity, and force, the phrase "in a certain direction" is the important part. Displacement without direction is simply distance; velocity without direction is simply speed; and there is no special term for force without direction. These definitions are taken from the study of physics.

Since distance + direction = displacement and speed + direction = velocity, we can say that distance is simply the magnitude of the vector that represents displacement, and speed is the magnitude of the vector that represents velocity.

### Directed Line Segments, Equal Vectors, and Absolute Value

Drawing the vector in example 1 is probably a good idea, as is reviewing the distance formula. Drawing a right triangle with the given vector as the hypotenuse will remind students of how the distance formula is derived from the Pythagorean Theorem. When students use the distance formula throughout the rest of this chapter, drawing right triangles will help them to remember it, but it is also a good idea for them to try to learn the formula well enough to use it without consulting diagrams.

The direction of a vector, by the way, is defined as the angle made by the vector when it is placed in standard position.

### Vector Addition and Subtraction

You'll need to skip over the section on subtraction until after you've covered the section on addition just below it.

### Vector Addition

Not only can we not use the parallelogram method to find the sum of a vector and itself, we also can't use it to find the sum of a vector and its opposite (and in the latter case, the tip-to-tail method doesn't work either.) However, the sum of a vector and its opposite is just 0, and the sum of a vector and itself is just a vector with twice the magnitude in the same direction.

### Resultant of Two Displacements

When we return to the problem about the ship, you may need to remind students that the magnitude of each vector represents the speed the ship is traveling in that direction, and therefore that the total speed is represented by the magnitude of the resultant vector.

In the balloon example, angle $A$ is the "bottom corner" of the triangle in the diagram. There is an easier way to find the angle with the horizontal, though, and you might ask students what it is. (Answer: The "top right corner" of the triangle is the angle we are looking for, and its tangent is $\frac{13}{22}$, from which we can calculate the angle directly.)

In case anyone is confused by part c of the "other things to consider" section, the phrase "(22ft/second times 14, 400 seconds in two hours)" should read "four hours." The numbers are still correct.

**Points to Consider**

When we add vectors using the triangle method, we know their lengths and can figure out the angle between them; hence we can use the Law of Cosines to figure out the magnitude and direction of the resultant vector.

This is a little less straightforward than it sounds, though, because we can't just read off the angle measures easily. Consider the following example, where $A$ and $B$ represent the directional angles of the vectors a and b that are being added by the triangle method:
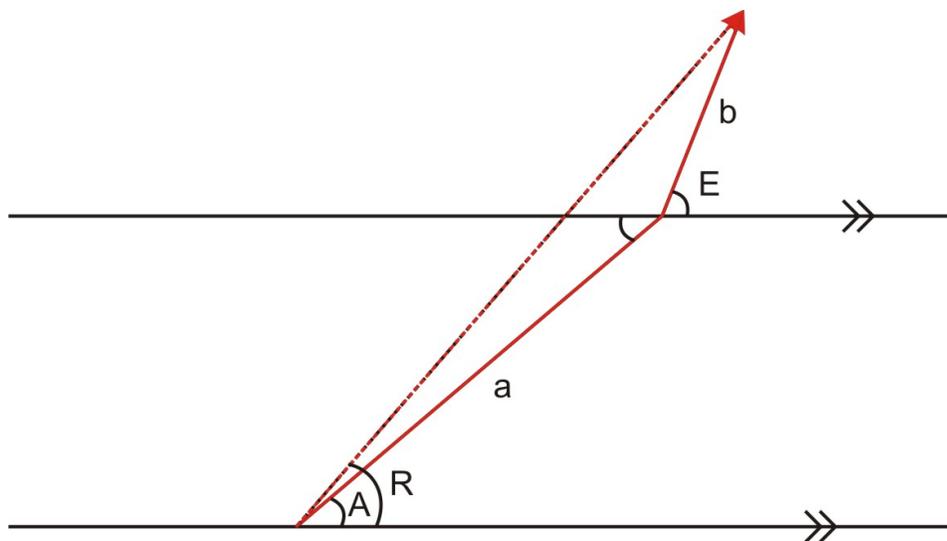


Unfortunately, $A$ and $B$ are the angles the vectors make with a horizontal line, not the angles within the triangle which we need to know in order to use the Law of Cosines—so first we need to use geometry to determine the angles within the triangle. We can see that the largest angle is equal to $A$ (by alternate interior angles) plus the supplement of $B$, and since we know sides $a$ and $b$, we can use the Law of Cosines to find the magnitude of the resultant vector. Then the Law of Sines will tell us the other angles of the triangle, from which we can figure out angle $R$, the direction angle of the resultant vector.

## Component Vectors

**Vector Times a Scalar**

Remind students here that $|\vec{a}|$ represents the magnitude of vector $\vec{a}$.

It may be worth explaining why we can "scale" a vector up or down just by multiplying each of its coordinates by the same scalar. Referring back to the methods from the previous lesson will show why this works: if you take the formula for finding the magnitude of the original vector, and multiply all the coordinates by the constant $k$, the result will be $k$ times the original magnitude; and if you use trig ratios to find the direction of the vector, those ratios will stay the same (and thus the direction stays the same) if you multiply all the coordinates by $k$.

**Translation of Vectors and Slope**

To multiply a vector by a scalar, as in Example 5, we don't actually need to translate it to the origin first; we can just multiply the coordinates of the initial and terminal points of the vector by the scalar k. However,

this problem provides useful practice in translating vectors as well as in scalar multiplication.

You may want to clarify that although the same ordered pair can represent many different vectors, it can only represent one vector in standard position. That vector is equivalent to, and in a sense represents, infinitely many other vectors with the same magnitude and direction but different initial points.

### Unit Vectors and Components

(Note: $\vec{i}$ and $\vec{j}$ are typically read as "$i$−hat" and "$j$−hat," but often you can call them simply "$i$" and "$j$" without confusing anyone.)

### Resultant as the Sum of Two Components

You may need to slow down to explain the notation here. In the diagram, the blue vector is $r$ and the green vector is $s$, but we are expressing them in a slightly different way to emphasize their role as component vectors. Since vector $r$ is horizontal, it can be expressed as $|\vec{r}|\hat{i}$—that is, as the horizontal unit vector $\hat{i}$ multiplied by the scalar quantity "the magnitude of $r$," yielding a vector with the same magnitude as $r$ (which makes sense because it is $r$!) and in the same direction as the horizontal unit vector $\hat{i}$. By the same logic, vector s can be expressed as $|\vec{s}|\hat{j}$, because it has the magnitude of $s$ and is in the direction of $\hat{j}$.

### Points to Consider

1) One way to verify answers to an addition or subtraction problem is to resolve each vector into its components and then add or subtract the components separately.

2) Vectors often form oblique triangles when added, so many of the same solving techniques apply.

3) Working with vectors can be more difficult because they are expressed in terms of their relationship to the origin rather than their relationship to each other, so finding things like the angles between them may be harder.

4) Vectors are usually used to solve problems where force is being applied in different directions, as opposed to problems that simply deal with distances between objects.

5) Unit vectors can help us visualize distances on a coordinate grid, but we don't generally need to use them when we are not looking for such distances.

## Real-World Triangle Problem Solving

### Introduction

To clarify situation 1: the climber will need enough rope to reach in a straight line from the top of the wall to the point where his partner will be standing (i.e. the point where he is now), so we need to find that distance to determine the necessary amount of rope.

To clarify situation 2: the axles run lengthwise through the centers of the cylinders and stick out at each end, so the steel cable to hold the cylinders together needs only to run around all three axles, rather than around the outsides of the cylinders. (This will be made clearer by the diagram in the next section.) Also, two loops of cable will need to be wrapped around the axles—one at each end of the cylinders—so once we find the length of one loop of cable, we will need to double it.

### Represent Problem Situations as Triangle(s)

In the diagram for situation 3, make sure students realize that the label $\theta$ applies to the whole span of the angle between the resultant vector and the horizontal line, not just the angle between the resultant vector and the vector next to it.

**Make a Problem-Solving Plan**

In order to figure out what information they still need to solve each problem, students may first need to think about what tools they might use to solve it. Whether or not the triangle is a right triangle is especially important to consider, as right triangles can be solved with much easier methods.

**Choose Among All Available Tools**

In situation 1, note that we are rounding our answers to the nearest whole number of feet; this is reasonable since nothing more precise was specified. In situation 2, however, since we started off with values that were rounded to the nearest tenth, it is reasonable to round our answers to the nearest tenth as well.

**Points to Consider**

1) We could verify our answer in situation 1 using different trig ratios, or trig ratios of different sides, than we used to find the answer. In situation 2, we could use the Law of Cosines to verify the angles that we didn't find with the Law of Cosines in the first place; in situation 3, we could do that with the laws of both Sines and Cosines. Those would be less reliable methods, though, as we might get mixed up and simply perform the same operations we used to get the answers in the first place, thereby verifying nothing.

2) Checking your answer might not help if you verified the answer with the same methods you used to get the answer—or with those same methods in disguise.

3) The Law of Sines might be unreliable for checking angles because of the Ambiguous Case.

4) Some of the tools used to check them might have been used to solve them; also, applying the same tools in a different order might have worked.

# 1.6 Polar Equations and Complex Numbers

## Polar Coordinates

### *Polar Coordinates*

### Introduction

On example 1, you may need to stress that we are moving clockwise because the $\theta$-coordinate is negative, and that otherwise we would be moving counterclockwise.

### *Sinusoids of One Revolution*

### Introduction

An important thing to explain about graphing in polar coordinates is that $r$ is always the dependent variable and $\theta$ is the independent variable—so the equations we graph in polar coordinates will take forms like $r = \sin\theta$, or more complex expressions based on $\theta$, where $\theta$ is always an angle measure. The fact that $r$ is always written first in the ordered-pair representation of a point is a little counterintuitive, because we have previously been used to seeing the independent variable (which is usually $x$) written first.

Why do we write $r$ first if $r$ is the dependent variable? One reason is that graphing points is easier, or at least more intuitive, if we look at the $r-$coordinate first; we can think of ourselves as starting at the origin, moving along the $x-$axis to the given $r-$value, and then moving around the circle to the given $\theta-$value.

(Another reason has to do with the conventions for representing complex numbers in polar form, which will be covered later in this chapter.)

Example 3 contains an interesting optical illusion: the diagonal lines passing through the graph make it look slightly warped and may prevent students from realizing that it is in fact a perfect circle.

Also worth pointing out about this example is that all of the points on the graph have in fact been traced twice over. The $\theta-$values from $0°$ to $180°$ traced out the circle; then the $\theta-$values from $180°$ to $360°$ produced the same set of $r-$values all over again, but negative. Since $(-r, \theta + 180°)$ always represents the same point as $(r, \theta)$, the second set of $r-$values correspond to the same set of points as the first set.

The important thing to point out about cardioids is that they are dimpled limaçons whose "dimple" passes directly through the pole. The depth of the dimple in a limaçon depends on the ratio $\frac{a}{b}$ (the smaller the ratio, the deeper the dimple), and the limaçon becomes a cardioid when the ratio equals 1. If the ratio got any smaller, the limaçon would dimple so far that it would develop a loop.

### Applications, Trigonometric Tools

### Introduction

The constraints on example 1 should read "$-2\pi \leq \theta \leq 2\pi$" rather than "$0 \leq \theta \leq 2\pi$".


# Polar-Cartesian Transformations

### Graphs of Polar Equations

Note that when we graph a basic cosine equation in polar coordinates, the domain only needs to go from 0 to $\pi$ rather than $2\pi$. As noted in the previous lesson, the sine or cosine graph is traced out twice over on an interval of $2\pi$ units. In a sense, we can almost say that when we are using polar coordinates, the sine and cosine functions have a period of $\pi$ instead of $2\pi$.

### Conic Section Transformations

### Introduction

An ellipse is actually the result of the intersection of a cone on *one* side by a plane that may or may not be parallel to the base of the cone. When the plane is parallel to the base, the ellipse is a circle.

The plane that creates a parabola cannot just be non-parallel to the base; it must be parallel to the slanted line that forms the edge of the cone.

The plane that creates a hyperbola does not actually have to be perpendicular to the base, as long as it intersects both halves of the cone.

Also, you may need to clarify that the definition of "cone" used here differs from the one students encountered in geometry: a "cone" here is really two cones lined up tip to tip, and the two cones actually extend infinitely far outward from the point where they meet.

Parabolas can theoretically be stretched horizontally as well as vertically, but stretching them horizontally is in a sense the same as shrinking (or "un-stretching") them vertically, so it can be expressed in the same way, with a little adjusting of arbitrary constants.

The focal axis of an ellipse is also called the major axis, and its length is denoted as 2a. The perpendicular line passing through the center of the ellipse is the minor axis and its length is 2b. This will be important later.

### Points to Consider

Circles centered at the origin are certainly easier to express in polar coordinates, but those that have been shifted away from the origin may be a little harder. Parabolas tend to be easier to represent in rectangular

coordinates. In general, taking a familiar equation and shifting or stretching it in one direction or another tends to be easier when the equation is expressed in rectangular coordinates. (These are just a few examples; students may provide others.)

Polar curves may of course intersect, as we saw during this lesson. This question prepares students for the next lesson, where they will learn to find the intersection points of such curves.

Since two different sets (in fact, infinitely many sets) of coordinates can be used to represent the same set of points, it makes sense that two different equations could be used to represent the same polar curve. For example, $r = \sin \theta$ would produce the same graph as $r = \sin(\theta + 2\pi)$.

One important difference between rectangular and polar representation is that polar graphs are more likely to stay within a finite viewing space. When we graph a function on a rectangular grid, if the function's domain is unlimited, then the graph extends infinitely far to each side, so we can't ever really draw the entire graph. Polar graphs, on the other hand, can extend infinitely far outward if the range is unlimited, but if the range is limited, then the domain can be unlimited and the graph will still be conveniently compact.

(Then again, this can be an inconvenience at times, as it makes it harder to show when we are deliberately only graphing part of a function instead of the whole thing. Since the whole thing *should* fit in the visible part of the graph, viewers will expect the visible part of the graph to contain the whole function unless we include a note specifying otherwise. With rectangular coordinates, we can simply narrow the graphing window to show only the part we want to graph, and viewers don't need to be told that there's really more to the graph than just that part.)

## Systems of Polar Equations

### *Graph and Calculate Intersections of Polar Curves*

### Introduction

In the solution to example 2, the notation $k\epsilon I$ may be unfamiliar to students; it means "$k$ is an element (a member) of the set of all integers," or simply "$k$ is an integer." So saying that the solution set includes $\left(1, \frac{5\pi}{3}\right) + 2\pi k, k\epsilon I$ is simply another way of saying that when we add any integer multiple of $2\pi$ to the $\theta-$coordinate in the solution $\left(1, \frac{5\pi}{3}\right)$, we get another valid solution—and of course the same holds true for the solution $\left(1, \frac{\pi}{3}\right)$.

The solution to Example 3 may be confusing at first—how can $(0, 0)$ and $\left(0, \frac{\pi}{2}\right)$ represent the same point? Students should grasp by now that adding a multiple of $2\pi$ to the $\theta-$coordinate of a point yields another representation of the same point, but in this case we've added $\frac{\pi}{2}$, which isn't a multiple of $2\pi$—so what's going on here?

The important thing to explain here is that the pole (as we call the origin when we are using polar coordinates) is a very special point. Normally, any given $r-$coordinate designates a circle centered at the pole with radius $r$, and we use different $\theta-$coordinates to pick out specific points on the circle. But a circle with radius 0 is just a single point—the pole itself—so no matter what $\theta-$coordinate we choose, we always end up at that same point. $(0, 0)$ represents the same point as $(0, 3), \left(0, \frac{\pi}{2}\right), (0, 4\pi)$, or any ordered pair whatsoever that has 0 as the $r-$coordinate.

### Points to Consider

A very simple example of two polar curves that do not intersect is the pair $r = 1$ and $r = 2$. And we have seen again in this lesson how the same point can be expressed in more than one way in polar coordinates; in the next lesson we will see how the same curve can too.

### *Equivalent Polar Curves*

It is important that students do not get the mistaken idea that expressions with equivalent graphs are necessarily equivalent expressions. In example 1b, the two equations graphed are indeed equivalent, as they are both simply different ways of expressing $r = 5$. But in example 1a, although the two equations trace out the same graph, they do not actually have the same $r-$value for any given $\theta-$value, and so are not equivalent equations. If they were equivalent, plugging the same $\theta-$value into both of them would always yield the same $r-$value.

This is especially confusing because it only happens in polar coordinates, where the $\theta$-values overlap and repeat themselves. Here's an analogy that may help you to explain it: Suppose you ride the same bus to work or school every day, and suppose the bus maintains a very strict schedule, so it always reaches the same stop on its route at exactly the same time each day. (Of course no real bus could manage this, but let's assume it does in order to simplify the problem.) Now suppose you draw a graph each day representing the route the bus travels, and when you compare two consecutive graphs, you see that they look exactly the same—if you plotted them on the same axes, they would look like just one graph. But does this mean they are describing the exact same trip? No—they represent two different trips taken on two different days, and when you plot them on the same axis you are simply leaving out the "two different days" part. Really, the time-values of the second graph are the time-values of the first graph "plus 24 hours," and if you choose to plot them on the same graph to save space, you must still remember that the times on the two graphs aren't "really" the same.

And that's what happens when we use polar coordinates—we sometimes end up graphing the $\theta-$values for two or more different $r-$values in what looks like the same spot, but we must remember that just because they are sharing space, that doesn't mean they are really the same coordinates. Even when a whole graph looks the same as another, sometimes it simply consists of a different set of values that happen to be graphed in the same places.

## Imaginary and Complex Numbers

### *Recognize*

### Introduction

Here's another example of why the rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$ only applies if $a$ and $b$ are not both negative: Without that exception, we could apply the rule to $\sqrt{36}$ and express it as $\sqrt{-4}\sqrt{-9}$, which would equal $2i$ times $3i$, or $-6$. Technically, $(-6)^2$ is of course 36, but officially -6 is not the square root of 36, so that answer would be incorrect.

The last line of the solution to example 2b should have a 5, rather than a 3, under the radical sign.

### Points to Consider

Students needn't know the answers to these questions; they are simply a preview of the next section.

### *Standard Form of Complex Numbers (a + bi)*

### Introduction

Students may be a little confused by the statement that a and b are both real numbers in the standard form $a + bi$. Clarify if necessary that the imaginary part of a complex number is $bi$, not just $b$; $bi$ is a pure imaginary number because it is a real number multiplied by $i$.

Students may not quite see why the answer to example 1c is expressed as it is. The answer is indeed in standard rectangular form, with $a = 3\sqrt{2}$ and $b = -2\sqrt{2}$, but we traditionally put the $i$ in front of the radical sign so that it doesn't look like it is included under the radical sign, and that makes it harder to see that the whole expression has the form $a + bi$.

After reviewing example 3, you might also want to challenge students to find the conjugate of a real number, like 5. (Answer: 5 is really $5 + 0i$, so its conjugate is $5-0i$, or simply 5 again. In other words, the conjugate of any real number is simply itself.)

**Points to Consider**

We will see in the next lesson what operations can be performed on complex numbers and with what results.

### *Complex Number Plane*

If you've covered vectors recently, you might point out here that the absolute value of the complex number $a + bi$ is the same as the magnitude of the vector represented by the point $(a, b)$.

You may need to skip over the problem about two students walking home, since the original formulation of the problem is missing from the lesson.

# Operations on Complex Numbers

### *Quadratic Formula*

**Points to Consider**

When the roots of an equation are complex, we know that the graph of the equation does not intersect the $x-$axis. Conversely, when the graph does not intersect the $x-$axis, we know the roots are complex, and when it does, we know there are either two real roots or one real root repeated twice.

### *Sums and Differences of Complex Numbers*

**Points to Consider**

We will see in the next lesson how complex numbers can be expressed in polar form.

### *Products and Quotients of Complex Numbers (conjugates)*

**Introduction**

There is a typographical error in the formula for multiplying complex numbers: the term that reads $(ad-bd)$ should read $(ac - bd)$.

Students may not quite see where the "$-bd$" term comes from. Explain that multiplying $bi$ and $di$ yields $bdi^2$, and $i^2$ is simply $-1$, leaving us with $-bd$.

The procedure for dividing complex numbers may make more sense if you remind students that $i$ is equal to $\sqrt{-1}$. When we express the quotient of two complex numbers as a fraction, substituting $\sqrt{-1}$ for $i$ shows that this fraction essentially has a radical in the denominator which we must rationalize. Multiplying the numerator and denominator by the conjugate of the denominator, then, is clearly the way to get the imaginary part out of the denominator; and once the denominator is a real number, we can divide both parts of the numerator by that real number and thus express the answer in standard form.

**Points to Consider**

Once again, we will see in the next lesson that the answer to both of these questions is "yes."

### *Applications, Trigonometric Tools*

**Operations on Complex Numbers**

Example 2 provides an excellent illustration of the relationship between operations on complex numbers and operations on vectors. You may want to point out to students that when they solve a problem like this

by working with the real and imaginary parts of complex numbers separately, they are really just resolving vectors into horizontal and vertical components and working with each component separately. But instead of using $\hat{i}$ and $\hat{j}$ to represent those components, they are using 1 and $i$, because the horizontal component is a multiple of the unit vector in the real direction or "1−direction" and the vertical component is a multiple of the unit vector in the imaginary or "$i$–direction."

# Trigonometric Form of Complex Numbers

### Trigonometric Form of Complex Numbers: Relationships among $x, y, r$, and $\theta$

In case students don't immediately see why $x = r\cos\theta$ and $y = r\sin\theta$, you can show them fairly easily on the diagram that $\sin\theta = \frac{y}{r}$, and then solving for $y$ yields $y = r\sin\theta$. Similar reasoning, of course, holds for $x$.

### The Trigonometric or Polar Form of a Complex Number $(rcis\theta)$

The term "cis$\theta$" is easier to remember if you point out that it is somewhat like an acronym, derived from "$\cos\theta + i\sin\theta$."

The term "argument" is also worth explaining, as it often appears elsewhere in mathematics. Generally it refers to the "input" of a given function, so for example, in the expression "$\cos\theta$," $\theta$ is called the argument of the cosine function. In the polar form of a complex number, of course, $\theta$ appears as the argument of both the sine and cosine functions, so it makes a kind of sense to call it the "argument" of the complex number as a whole.

Thinking of r as the "absolute value" of a complex number may be counterintuitive for students, but it is really just an extension of the idea of absolute value of real numbers. The absolute value of a real number is its distance from 0; the absolute value of a complex number is its distance from the point $(0,0)$. And in the complex plane, the distance of a real number from 0 becomes the same thing as its distance from $(0,0)$.

### Trigonometric Form of Complex Numbers: Steps for Conversion

#### Introduction

The very first table in this section is the most useful for students to know, and it is most helpful for them to think of the left half and the right half separately. Emphasize that if they know the polar coordinates $r$ and $\theta$, they should use the two equations on the *left* to find the rectangular coordinates $x$ and $y$, whereas if they know $x$ and $y$ they should use the two equations on the *right* to find $r$ and $\theta$. (They *could* use the equations on the right to get $x$ and $y$ from $r$ and $\theta$, or use the equations on the left to get $r$ and $\theta$ from $x$ and $y$, but that would be a much messier process.)

The last step of Example 3 is important: finding the inverse tangent just tells us the reference angle for $\theta$, not $\theta$ itself. We must then apply our knowledge of what quadrant the complex number is in to figure out what angle $\theta$ really is. We can find out what quadrant the number is in by graphing the rectangular coordinates we started out with, or by simply noting the signs of those coordinates: $x$ is only positive in the first two quadrants, and $y$ is only positive in the first and fourth.

#### Points to Consider

In the next lesson, we will see how to perform basic operations such as multiplication and division on complex numbers in polar form.

## Product and Quotient Theorems

### Product Theorem

The first line of the derivation here uses the FOIL method for multiplying binomials; in the second line, we group together the terms with $i$ in them and factor out the $i$; and in the third line, we apply the angle sum rules for sine and cosine in reverse.

The Product Theorem may be easier for students to remember if summarized in words: "To multiply complex numbers in polar form, multiply the $r-$coordinates and add the $\theta-$coordinates."

### Quotient Theorem

Similarly, the Quotient Theorem can be summarized: "To divide complex numbers in polar form, divide the $r-$coordinates and subtract the $\theta-$coordinates." The derivation follows much the same procedure as the one for the product theorem; note that the denominator of the final fraction equals 1, so we can cancel it out.

Incidentally, some students may notice that we haven't discussed how to add and subtract complex numbers in polar form. It turns out that there is no handy formula for doing so; the only good way to add and subtract complex numbers is to convert them to rectangular form first.

### Using the Quotient and Product Theorem

#### Introduction

Students may be confused by example 3, or may be tempted to do the division on the numbers as given, in rectangular form. Doing it as the book suggests, however, will give them practice in converting from rectangular to polar form as well as in performing division on numbers in polar form. Meanwhile, the other three examples will give them practice in working with complex numbers expressed in polar form in more than one way.

For extra practice, you might have them find both the product and the quotient of the two numbers in each example, instead of just the product or the quotient.

#### Points to Consider

So far, we've actually just barely touched on squares and square roots of complex numbers, and we still don't know how to find them for most numbers. The next lesson will cover these as well as other powers and roots of complex numbers.

### Applications and Trigonometric Tools: Real-Life Problem

You may need to skip these problems, as they contain terms that students are not likely to know .

## Powers and Roots of Complex Numbers

### De Moivre's Theorem

A more intuitive way to express De Moivre's Theorem is "To raise a complex number to the $n^{\text{th}}$ power, raise the $r-$coordinate to the $n^{\text{th}}$ power and multiply the $\theta-$coordinate by $n$."

In example 2, the expression $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right)$ should read $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)$.

#### nth Root Theorem

Here's a much more intuitive way to explain the $n^{\text{th}}$ Root Theorem:

Every complex number has exactly $n$ $n^{\text{th}}$ roots, which are evenly spaced around a circle in the complex plane.

If the original number has coordinates $(r, \theta)$, then the first of the $n^{\text{th}}$ roots (which, incidentally, is known as the *principal root*) has coordinates $\left( \sqrt[n]{r}, \frac{\theta}{n} \right)$. The rest of the $n^{\text{th}}$ roots all have the same $r$−coordinate, and their $\theta$−coordinates are each $\frac{\theta}{n}$ plus some multiple of $\frac{2\pi}{n}$; in other words, each of them is $\frac{1}{n}$ of the way around the circle from the one before it.

For example, the fourth roots of $(16, 60°)$ are $(2, 15°), (2, 105°), (2, 195°)$, and $(2, 285°)$. Plotting these points shows that they are evenly spaced around a circle of radius 2, and a little thought will show why. Raising 2 to the fourth power of course gives us 16, and multiplying an angle of $15°$ by 4 gives us $60°$ —but multiplying an angle of $15°$-plus-some-multiple-of-$90°$ by 4 gives us $60°$-plus-some-multiple-of-$360°$, which is equivalent to $60°$. So that's why, if $(2, 15°)$ is one fourth root of $(16, 60°)$, all the other fourth roots are of the form $(2, 15° + k * 90°)$ for some integer $k$. (If they were fifth roots, they'd be $\frac{360°}{5}$ or $72°$ apart; if they were sixth roots they'd be $60°$ apart, and so on.)

Incidentally, we could just keep going, adding $90°$ to each previous $\theta$−coordinate to get yet another one. But of course, after we've added $90°$ three times to the first solution to get three more solutions, adding $90°$ one more time would just give us the first solution plus $360°$, or in other words the first solution all over again, and then we'd start cycling through all the solutions over again. So the first four solutions (the first $n$ solutions, if we're finding $n^{\text{th}}$ roots) are the only unique ones, and we can stop after finding them.

### Solve Equations

It should become clear here that we can use polar coordinates to determine the $n^{\text{th}}$ roots of a pure real number or pure imaginary number just as easily as a complex number. However, the roots are usually complex and tend to be somewhat messy to express in rectangular coordinates. For example, in the problem shown in the text, we can see that the roots can be expressed precisely in polar coordinates, and that we can use just one expression to summarize them all, whereas in rectangular coordinates we must list all the roots separately and can only express them in decimal approximations.